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Sobolev orthogonal functions on the grid, generated by discrete orthogonal functions and the Cauchy problem for the difference equation

We consider the system of functions $\psi_{r,n}(x)$ $(r = 1, 2, \ldots, n = 0, 1, \ldots)$ orthonormal on Sobolev with respect to the inner product of the form $\langle f, g\rangle = \sum_{k=0}^{r-1} \Delta^k f(0)\Delta^k g(0) + \sum_{j=0}^{\infty} \Delta^r f(j)\Delta^r g(j)\rho(j)$, generated by a given orthonormal system of functions $\psi_n(x)$ $(n = 0, 1, \ldots)$. It is shown that the Fourier series and Fourier sums by the system $\psi_{r,n}(x)$ $(r = 1, 2, \ldots, n = 0, 1, \ldots)$ are convenient and a very effective tool for the approximate solution of the Cauchy problem for difference equations.

Bibliography: 19 items.

Рассмотрены системы функций $\psi_{r,n}(x)$ $(r = 1, 2, \ldots, n = 0, 1, \ldots)$, ортонормированные по Соболеву относительно скалярного произведения вида $\langle f, g\rangle = \sum_{k=0}^{r-1} \Delta^k f(0)\Delta^k g(0) + \sum_{j=0}^{\infty} \Delta^r f(j)\Delta^r g(j)\rho(j)$, порожденные заданной ортонормированной системой функций $\psi_n(x)$ $(n = 0, 1, \ldots)$. Показано, что ряды и суммы Фурье по системе $\psi_{r,n}(x)$ $(r = 1, 2, \ldots, n = 0, 1, \ldots)$ являются удобным и весьма эффективным инструментом приближенного решения задачи Коши для разностных уравнений.

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Ключевые слова: функции, ортогональные по Соболеву, функции, ортогональные на сетке, приближение дискретных функций, смешанные ряды по функциям, ортогональным на равномерной сетке, итерационный процесс для приближенного решения разностных уравнений.

Introduction

We continue the consideration of systems of discrete functions begun in [1] and [2], orthogonal with respect to the Sobolev-type inner products of the following form

$$\langle f, g\rangle = \sum_{k=0}^{r-1} \Delta^k f(0)\Delta^k g(0) + \sum_{j=0}^{\infty} \Delta^r f(j)\Delta^r g(j)\rho(j), \qquad (0.1)$$

where functions f and g are defined on the set $\Omega = \{0, 1, \ldots, \}$, $\rho = \rho(j)$ $(j \in \Omega)$ – discrete weight function. In the case when $r = 0$ we assume that

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$\sum_{k=0}^{r-1}\Delta^k f(0)\Delta^k g(0) = 0$. For $r \geqslant 1$, the singular point in the inner product (0.1) is $x = 0$, in vicinity of which it is «controlled» the behavior of corresponding functions, orthogonal on Sobolev. This is achieved by presence in the inner product (0.1) of the expression $\sum_{k=0}^{r-1}\Delta^k f(0)\Delta^k g(0)$. The interest to similar systems of functions intensive growing lately (see [1] – [18] and the literature cited there). This is explained, among other things, by the fact that Sobolev orthogonal functions turned out to be a convenient tool for solving boundary-value problems of differential and difference equations. In this work (see Section 2) we show that systems of discrete functions orthonormal on Sobolev can be used for approximate the solution of difference equations (including nonlinear ones) in the form of series by functions orthogonal on Sobolev. The main idea of the approach under consideration is to construct an iterative process for the approximate determination of the unknown coefficients of the indicated decomposition of the desired solution of the difference equation. In proving the convergence of the constructed iterative process, a crucial role is played by the properties of functions orthonormal on Sobolev and generated by a given orthonormal system of functions $\psi_n(x)$ $(n = 0, 1, \ldots)$.

1. Systems of discrete functions orthonormal on Sobolev, generated by orthonormal system of functions

We need some notions and results from the work [1], in which systems of discrete functions orthonornal on Sobolev with respect to the inner product (0.1), generated by the system $\{\psi_k(x)\}_{k=0}^{\infty}$ orthonormal on the discrete set $\Omega = \{0, 1, \ldots\}$ with weight $\rho(x)$ are considered. For the integer $k \geqslant 0$ we assume that $a^{[k]} = a(a-1)\cdots(a-k+1)$, $a^{[0]} = 1$ and consider the following function

$$\psi_{r,k}(x) = \frac{x^{[k]}}{k!},\ k = 0, 1, \ldots, r-1, \tag{1.1}$$

$$\psi_{r,k}(x) = \begin{cases} \frac{1}{(r-1)!}\sum\limits_{t=0}^{x-r}(x-1-t)^{[r-1]}\psi_{k-r}(t), & r \leqslant x, \\ 0, & x = 0, 1, \ldots, r-1, \end{cases} \tag{1.2}$$

which are defined on the grid Ω. Consider some important difference properties of the system functions $\psi_{r,k}(x)$, defined by the equalities (1.1) and (1.2). We introduce the operator of finite difference Δf: $\Delta f(x) = f(x+1) - f(x)$ and put $\Delta^{\nu+1} f(x) = \Delta\Delta^{\nu} f(x)$. The following assertion, established in [1], we will cite together with its brief proof.

Lemma 1. *The following equalities hold*

$$\Delta^{\nu}\psi_{r,k}(x) = \begin{cases} \psi_{r-\nu,k-\nu}(x), & \text{if } 0 \leqslant \nu \leqslant r-1,\ r \leqslant k, \\ \psi_{k-r}(x), & \text{if } \nu = r \leqslant k, \\ \psi_{r-\nu,k-\nu}(x), & \text{if } \nu \leqslant k < r, \\ 0, & \text{if } k < \nu \leqslant r. \end{cases} \tag{1.3}$$

Proof. The truth of the lemma 1 when $r = 1$ is almost obvious, therefore we assume that $r \geqslant 2$. First of all note that if $f(x) = (x-1-t)^{[r-1]}$, then

$\Delta f(x) = (r-1)(x-1-t)^{[r-2]}$, therefore for $r \leqslant x, k$ due to (1.2) we have

$$\Delta\psi_{r,k}(x) = \frac{1}{(r-1)!}\left(\sum_{t=0}^{x-r+1}(x-t)^{[r-1]}\psi_{k-r}(t) - \sum_{t=0}^{x-r}(x-1-t)^{[r-1]}\psi_{k-r}(t)\right) =$$

$$\frac{1}{(r-1)!}\sum_{t=0}^{x-r+1}\left((x-t)^{[r-1]} - (x-1-t)^{[r-1]}\right)\psi_{k-r}(t) =$$

$$\frac{1}{(r-2)!}\sum_{t=0}^{x-r+1}(x-1-t)^{[r-2]}\psi_{k-1-(r-1)}(t) = \psi_{r-1,k-1}(x).$$

Hence we verify the validity of the first of equalities (1.3) for $r \leqslant x$. The validity of equality $\Delta\psi_{r,k}(x) = \psi_{r-1,k-1}(x)$ for $x = 0, 1, \dots, r-2$ is obvious. It remains to verify the first of the equalities (1.3) for $x = r-1$. But in this case we have

$$\Delta\psi_{r,k}(x) = \psi_{r,k}(x+1) = \frac{1}{(r-1)!}\sum_{t=0}^{x-r+1}(x-t)^{[r-1]}\psi_{k-r}(t) =$$

$$\frac{(r-1)^{[r-1]}}{(r-1)!}\psi_{k-r}(0) = \psi_{k-r}(0),$$

$$\psi_{r-1,k-1}(x) = \frac{1}{(r-2)!}\sum_{t=0}^{x-r+1}(x-1-t)^{[r-2]}\psi_{(k-1)-(r-1)}(t) =$$

$$\frac{(r-2)^{[r-2]}}{(r-2)!}\psi_{k-r}(0) = \psi_{k-r}(0),$$

therefore we find again $\Delta\psi_{r,k}(x) = \psi_{r-1,k-1}(x)$. Thus, the first of the equalities (1.3) for $0 \leqslant \nu \leqslant 1$ is completely proved. Its validity for $2 \leqslant \nu \leqslant r-1$ we derive by induction. Consider the second of the equalities (1.3). Due to the proved the first equality (1.3) and the fact that the second of them for $r = 1$ is easily verified, we have

$$\Delta^r\psi_{r,k}(x) = \Delta\Delta^{r-1}\psi_{r,k}(x) = \Delta\psi_{1,k-r+1}(x) = \psi_{k-r}(x).$$

Thus we proved the second of the equalities (1.3). The third and the fourth of the equalities (1.3) follow directly from (1.1). Lemma 1 is fully proved.

We will call the system of functions $\psi_{r,k}(t)\,(k = 0, 1, \dots)$ following [1] *system, orthonormal on Sobolev with respect to the inner product* (0.1).

Let r – natural number. We denote by l_ρ the space of functions f, given on the grid $\Omega = \{0, 1, \dots\}$ such that $\sum_{x\in\Omega} f^2(x)\rho(x) < \infty$, $W^r_{l_\rho}$ is the space of discrete functions f, given on the grid Ω for which $f(x+\nu) \in l_\rho$, $\nu = 0, 1, \dots, r$. Then in the space $W^r_{l_\rho}$ the inner product $\langle f, g\rangle$ we define using the equation (0.1). We consider the problem of orthogonality, normality and completeness in $W^r_{l_\rho}$ of system $\{\psi_{r,k}(x)\}_{k=0}^{\infty}$, consisting of functions, defined by the equalities (1.1) and (1.2).

Theorem 1. *Suppose that functions $\psi_k(x)$ $(k = 0, 1, \dots)$ form a complete in l_ρ orthonormal system with weight $\rho(x)$. Then system $\{\psi_{r,k}(x)\}_{k=0}^{\infty}$, generated by the system $\{\psi_k(x)\}_{k=0}^{\infty}$ by the equalities* (1.1) *and* (1.2) *is complete in $W^r_{l_\rho}$ and orthonormal with respect to the inner product* (0.1).

PROOF. From equality (1.2) it is follows, that if $r \leqslant k$ and $0 \leqslant \nu \leqslant r-1$, then $\Delta^\nu \psi_{r,k}(x))_{x=0} = 0$ therefore due to (0.1) and (1.3) we have

$$\langle \psi_{r,k}, \psi_{r,l} \rangle = \sum_{x=0}^{\infty} \Delta^r \psi_{r,k}(x) \Delta^r \psi_{r,l}(x) \rho(x) =$$

$$\sum_{x=0}^{\infty} \psi_{k-r}(x) \psi_{k-l}(x) \rho(x) = \delta_{kl}, \quad k, l \geqslant r,$$

$$\langle \psi_{r,k}, \psi_{r,l} \rangle = \sum_{\nu=0}^{r-1} \Delta^\nu \psi_{r,k}(0) \Delta^\nu \psi_{r,l}(0) = \delta_{kl}, \quad k, l < r.$$

Also obvious, that

$$\langle \psi_{r,k}, \psi_{r,l} \rangle = 0, \quad \text{if} \quad k < r \leqslant l \quad \text{and} \quad l < r \leqslant k.$$

It means that functions $\psi_{r,k}(t)\,(k = 0, 1, \ldots)$ form in $W^r_{l_\rho}$ orthonormal system with respect to the inner product (0.1). To verify the completeness of this system in $W^r_{l_\rho}$, we assume that for functions $f \in W^r_{l_\rho}$ the following equalities hold

$$\langle \psi_{r,k}, f \rangle = 0, \quad k = 0, 1, \ldots.$$

Then, first, due to the fact that $0 = \langle \psi_{r,k}, f \rangle = \Delta^k f(0)$ for $k = 0, \ldots, r-1$ we have $f(j) = 0$ for all $j = 0, \ldots, r-1$. Second, from the equalities $\langle \psi_{r,k}, f \rangle = 0, \quad k = r, r+1, \ldots$ and the completeness in l_ρ of the original system $\psi_k(t)\,(k = 0, 1, \ldots)$ follows that $\Delta^r f(x) \equiv 0$ $(x \in \Omega)$ and therefore f coincides with the algebraic polynomial degree not higher than $r-1$. From these two facts follows that $f(x) \equiv 0$ $(x \in \Omega)$. The theorem is proved.

From Theorem 1 follows that system of discrete functions $\psi_{r,k}(t)\,(k = 0, 1, \ldots)$ is orthonormal basis (ONB) in the space $W^r_{l_\rho}$, therefore for the arbitrary function $f(x) \in W^r_{l_\rho}$ we can write the equality

$$f(x) = \sum_{k=0}^{\infty} \langle f, \psi_{r,k} \rangle \psi_{r,k}(x), \tag{1.4}$$

which is the Fourier series of the function $f(x) \in W^r_{l_\rho}$ by the system $\{\psi_{r,k}(t)\}_{k=0}^{\infty}$, orthonormal on Sobolev. Since the Fourier coefficients $\langle f, \psi_{r,k} \rangle$ are of the form

$$f_{r,k} = \langle f, \psi_{r,k} \rangle = \sum_{\nu=0}^{r-1} \Delta^\nu f(0) \Delta^\nu \psi_{r,k}(0) = \Delta^k f(0),\ k = 0, \ldots, r-1,$$

$$f_{r,k} = \langle f, \psi_{r,k} \rangle = \sum_{j=0}^{\infty} \Delta^r f(j) \Delta^r \psi_{r,k}(j) \rho(j) =$$

$$\sum_{j=0}^{\infty} \Delta^r f(j) \psi_{k-r}(j) \rho(j),\ k = r, \ldots,$$

then we can write the (1.4) in mixed form

$$f(x)=\sum_{k=0}^{r-1}\Delta^k f(0)\frac{x^{[k]}}{k!}+\sum_{k=r}^{\infty}f_{r,k}\psi_{r,k}(x). \tag{1.5}$$

Therefore Fourier series by the system $\{\psi_{r,k}(t)\}_{k=0}^{\infty}$ following [9] – [17] we will call mixed series by the original orthonormal system $\{\psi_k(t)\}_{k=0}^{\infty}$.

THEOREM 2. *Suppose that functions $\psi_k(x)$ $(k=0,1,\ldots)$ form a complete in l_ρ orthonormal system with weight $\rho(x)$. Then if $f(x)\in W^r_{l_\rho}$ the mixed series (1.5) converges pointwise $x\in\{0,1,\ldots\}$ to $f(x)$.*

PROOF. From the fact that $f(x)\in W^r_{l_\rho}$ follows that the function $g(t)=\Delta^r f(t)\in l_\rho$, therefore due to of the completeness in l_ρ of the system $\{\psi_k(x)\}$ there is an expansion in a series $g(t)=\sum_{k=0}^{\infty}f_{r,k+r}\psi_k(t)$, which converges in the metric of the space l_ρ, and hence, in turn, follows that this series converges pointwise $t\in\Omega$ to $g(t)$.

On the other hand, we use the discrete analog of the Taylor formula

$$f(x)=Q_{r-1}(f,x)+\frac{1}{(r-1)!}\sum_{t=0}^{x-r}(x-1-t)^{[r-1]}\Delta^r f(t), \tag{1.6}$$

where

$$Q_{r-1}(f,x)=f(0)+\frac{\Delta f(0)}{1!}x+\frac{\Delta^2 f(0)}{2!}x^{[2]}+\ldots+\frac{\Delta^{r-1}f(0)}{(r-1)!}x^{[r-1]}.$$

Substituting in (1.6) instead of the function $\Delta^r f(t)$ its expansion $g(t)=\sum_{k=0}^{\infty}f_{r,k+r}\psi_k(t)$. This gives

$$f(x)=Q_{r-1}(f,x)+\frac{1}{(r-1)!}\sum_{t=0}^{x-r}(x-1-t)^{[r-1]}\sum_{k=0}^{\infty}f_{r,k+r}\psi_k(t).$$

Hence, by interchanging summation signs, we find

$$f(x)=Q_{r-1}(f,x)+\sum_{k=0}^{\infty}f_{r,k+r}\frac{1}{(r-1)!}\sum_{t=0}^{x-r}(x-1-t)^{[r-1]}\psi_k(t)=$$

$$Q_{r-1}(f,x)+\sum_{k=0}^{\infty}f_{r,k+r}\psi_{r,k+r}(x)=\sum_{k=0}^{r-1}\Delta^k f(0)\frac{x^{[k]}}{k!}+\sum_{k=r}^{\infty}f_{r,k}\psi_{r,k}(x),\quad x\in\Omega.$$

As was to be shown.

We note some important properties of mixed series (1.5) and their partial sums of the form

$$\mathcal{Y}_{r,n}(f,x)=\sum_{k=0}^{r-1}\Delta^k f(0)\frac{x^{[k]}}{k!}+\sum_{k=r}^{n}f_{r,k}\psi_{r,k}(x). \tag{1.7}$$

From (1.5) and (1.7) taking into account equalities (1.3) we may write ($0 \leqslant \nu \leqslant r-1$, $x \in \Omega$)

$$\Delta^\nu f(x) = \sum_{k=0}^{r-\nu-1} \Delta^{k+\nu} f(0) \frac{x^{[k]}}{k!} + \sum_{k=r-\nu}^{\infty} f_{r,k+\nu} \psi_{r-\nu,k}(x),$$

$$\Delta^\nu \mathcal{Y}_{r,n}(f,x) = \sum_{k=0}^{r-\nu-1} \Delta^{k+\nu} f(0) \frac{x^{[k]}}{k!} + \sum_{k=r-\nu}^{n-\nu} f_{r,k+\nu} \psi_{r-\nu,k}(x),$$

$$\Delta^\nu \mathcal{Y}_{r,n}(f,x) = \mathcal{Y}_{r-\nu,n-\nu}(\Delta^\nu f, x).$$

2. On the representation of solution of the difference equation as Fourier series by functions $\psi_{r,n}(x)$

In this section, we consider the problem of finding an approximate solution of a difference equation by Fourier sums on the system $\{\psi_{r,n}(x)\}_{n=0}^\infty$, orthonormal on Sobolev and generated by orthonormal system of function $\{\psi_n(x)\}_{n=0}^\infty$ derived by equalities (1.1) and (1.2). Similar method in case when functions $\{\psi_n(x)\}_{n=0}^{N-1}$ are orthogonal on finite grid $0, 1, ..., N-1$ was developed in paper [19]. The results obtained below (Theorem 3) can be generalized to systems of difference equations of the form $\Delta y(x) = hf(x,y)$, $\quad y(0) = y_0$, where $f = (f_1, \dots, f_m)$, $y = (y_1, \dots, y_m)$. But for simplicity of calculations, we consider a problem of the form ($\Delta y(x) = y(x+1) - y(x)$)

$$\Delta y(x) = hf(x,y), \quad y(0) = y_0, \quad h > 0, \tag{2.1}$$

in which the function $f(x,y)$ is assumed to be given on the Cartesian product $\Omega \times \mathbb{R}$ and bounded on it, that is $M(f) = \sup_{(x,y)\in\Omega\times\mathbb{R}} |f(x,y)| < \infty$. It is required to approximate with a given accuracy the function $y = y(x)$ defined on Ω, which is the solution of the problem (2.1).

We suppose that the system $\{\psi_n(x)\}_{n=0}^\infty$ satisfies the conditions of Theorem 1, and the generated system $\{\psi_{1,n}(x)\}_{n=0}^\infty$ to the condition

$$\delta_\psi(x) = \sum_{k=1}^{\infty} (\psi_{1,k}(x))^2 < \infty \quad (x \in \Omega), \tag{2.2}$$

the positive weight function $\rho(x)$ and some given nonnegative function $\gamma(x)$ satisfy conditions

$$\sum\nolimits_{x=0}^{\infty} \rho(x) < \infty, \quad \kappa_\psi = \left(\sum_{t=0}^{\infty} \sum_{k=1}^{\infty} (\psi_{1,k}(t))^2 \rho(t) \gamma(t) \right)^{\frac{1}{2}} < \infty. \tag{2.3}$$

Since, by hypothesis, the function $f(x,y)$ is bounded on the set $\Omega \times \mathbb{R}$, then from the equality (2.1) and the first inequality (2.3) follows that

$$\sum_{x=0}^{\infty} (\Delta y(x))^2 \rho(x) < \infty$$

and, consequently, $y(x) \in W^1_{l_\rho}$. Therefore, by Theorem 2, we can represent $y(x)$ in the form of a convergent series

$$y(x) = y(0) + \sum\nolimits_{k=0}^{\infty} y_{1,k+1}\psi_{1,k+1}(x), \quad x \in \Omega. \tag{2.4}$$

where

$$y_{1,k+1} = \sum_{t=0}^{\infty} \Delta y(t)\psi_k(t)\rho(t) \quad (k \geqslant 0) \tag{2.5}$$

– the Fourier coefficients of the function $g(t) = \Delta y(t)$ by the system $\{\psi_k(x)\}$. Given (2.5), we can write

$$\Delta y(x) = \sum\nolimits_{k=0}^{\infty} y_{1,k+1}\psi_k(x), \quad x \in \Omega.$$

Moreover, taking (2.1) into account, the last equality can be written in such form

$$q(x) = f(x, y(x)) = \frac{1}{h}\Delta y(x) = \sum\nolimits_{k=0}^{\infty} c_k(q)\psi_k(x), \quad x \in \Omega,$$

where

$$c_k(q) = \frac{1}{h}y_{1,k+1} = \sum_{t=0}^{\infty} f(t, y(t))\psi_k(t)\rho(t) \quad (k \geqslant 0). \tag{2.6}$$

Using (2.6), we can rewrite the equality (2.4) as follows

$$y(x) = y(0) + h\sum\nolimits_{k=0}^{\infty} c_k(q)\psi_{1,k+1}(x), \quad x \in \Omega. \tag{2.7}$$

In the equality (2.6) we replace $y(t)$ by (2.7) and write

$$c_k(q) = \sum_{t=0}^{\infty} f\left[t, y(0) + h\sum\nolimits_{j=0}^{\infty} c_j(q)\psi_{1,j+1}(t)\right]\psi_k(t)\rho(t) \quad k = 0, 1, \dots. \tag{2.8}$$

To find approximate values of the coefficients $c_k(q)$ $(k = 0, 1, \dots)$ we construct an iterative process. We introduce the Hilbert space l_2 consisting of the sequences $C = (c_0, c_1, \dots)$ for which $\|C\| = \left(\sum_{j=0}^{\infty} c_j^2\right)^{\frac{1}{2}}$. In the space l_2 consider the operator A, which matches a point $C \in l_2$ to point $C' \in l_2$ by the rule

$$c_k' = \sum_{t=0}^{\infty} f\left[t, y(0) + h\sum\nolimits_{j=0}^{\infty} c_j\psi_{1,j+1}(t)\right]\psi_k(t)\rho(t) \quad k = 0, 1, \dots. \tag{2.9}$$

From (2.8) it follows that the point $C(q) = (c_0(q), c_1(q), \dots)$ is a fixed point of the operator $A : l_2 \to l_2$. In order to find this point by the fixed point iteration method, it suffices to show that the operator $A : l_2 \to l_2$ is a contraction in the metric of l_2. To this end we consider two points $P, Q \in l_2$, where $P = (p_0, p_1, \dots)$, $Q = (q_0, q_1, \dots)$ and let us assume $P' = A(P)$, $Q' = A(Q)$. We have

$$p_k' - q_k' = \sum_{t=0}^{\infty} F_{P,Q}(t)\psi_k(t)\rho(t)dt, \quad k = 0, 1, \dots \tag{2.10}$$

where

$$F_{P,Q}(t) = f\left[t, y(0) + h\sum\nolimits_{j=0}^{\infty} p_j\psi_{1,j+1}(t)\right] \\ - f\left[t, y(0) + h\sum\nolimits_{j=0}^{\infty} q_j\psi_{1,j+1}(t)\right]. \quad (2.11)$$

From (2.10), using Bessel's inequality, we find

$$\sum\nolimits_{k=0}^{\infty}(p'_k - q'_k)^2 \leqslant \sum_{t=0}^{\infty}(F_{P,Q}(t))^2\rho(t). \quad (2.12)$$

Suppose that the function $f(x, y)$ satisfies the Lipschitz condition by the variable y

$$|f(x, y') - f(x, y'')| \leqslant \sqrt{\gamma(x)}\lambda|y' - y''|, \quad x \in \Omega, \quad (2.13)$$

where $\gamma(x)$ – function for which the inequality (2.3) holds. From (2.11) and (2.13) we have

$$(F_{P,Q}(t))^2 \leqslant (\lambda h)^2\gamma(t)\left(\sum\nolimits_{j=0}^{\infty}(p_j - q_j)\psi_{1,j+1}(t)\right)^2,$$

from which, using the Cauchy–Bunyakovsky inequality and the condition (2.2), we derive

$$(F_{P,Q}(t))^2 \leqslant (\lambda h)^2\gamma(t)\sum\nolimits_{j=0}^{\infty}(p_j - q_j)^2\sum\nolimits_{j=0}^{\infty}(\psi_{1,j+1}(t))^2. \quad (2.14)$$

Gathering (2.14) with (2.12), we find

$$\sum\nolimits_{k=1}^{\infty}(p'_k - q'_k)^2 \leqslant (\lambda h)^2\sum\nolimits_{k=0}^{\infty}(p_k - q_k)^2\sum\nolimits_{t=0}^{\infty}\sum\nolimits_{j=0}^{\infty}(\psi_{1,j+1}(t))^2\rho(t)\gamma(t). \quad (2.15)$$

From (2.15) and (2.3) we have

$$\left(\sum\nolimits_{k=0}^{\infty}(p'_k - q'_k)^2\right)^{\frac{1}{2}} \leqslant \kappa_\psi\lambda h\left(\sum\nolimits_{k=0}^{\infty}(p_k - q_k)^2\right)^{\frac{1}{2}}. \quad (2.16)$$

The inequality (2.16) shows that if $\kappa_\psi\lambda h < 1$, then the operator $A : l_2 \to l_2$ is a contraction and as the consequence, the iterative process $C^{\nu+1} = A(C^\nu)$ converges to the point $C(q)$ as $\nu \to \infty$. However, in applications it is important to consider the finite-dimensional analogue of the operator A. We consider the operator $A_N : \mathbb{R}^N \to \mathbb{R}^N$, which matches a point $C_N = (c_0, \dots, c_{N-1}) \in \mathbb{R}^N$ to point $C'_N = (c'_0, \dots, c'_{N-1}) \in \mathbb{R}^N$ by the rule

$$c'_k = \sum_{t=0}^{\infty} f\left[t, y(0) + h\sum\nolimits_{j=0}^{N-1} c_j\psi_{1,j+1}(t)\right]\psi_k(t)\rho(t), \quad k = 0, 1, \dots, N-1. \quad (2.17)$$

Consider two points $P_N, Q_N \in \mathbb{R}^N$, where $P_N = (p_0, p_1, \dots, p_{N-1})$, $Q_N = (q_0, q_1, \dots, q_{N-1})$ and let us assume $P'_N = A_N(P_N)$, $Q'_N = A_N(Q_N)$. Repeating verbatim the arguments that led us to the inequality (2.16), we get

$$\left(\sum\nolimits_{k=0}^{N-1}(p'_k - q'_k)^2\right)^{\frac{1}{2}} \leqslant \kappa_\psi\lambda h\left(\sum\nolimits_{k=0}^{N-1}(p_k - q_k)^2\right)^{\frac{1}{2}}. \quad (2.18)$$

The inequality (2.18) shows that if $\kappa_\psi \lambda h < 1$, then the operator $A_N : \mathbb{R}^N \to \mathbb{R}^N$ is a contraction and as the consequence, the iterative process $C_N^{\nu+1} = A_N(C_N^\nu)$ converges to its fixed point, which we denote by $\bar{C}_N(q) = (\bar{c}_0(q), \dots, \bar{c}_{N-1}(q))$. On the other hand, consider the point $C_N(q) = (c_0(q), \dots, c_{N-1}(q))$, composed of the desired Fourier coefficients of the function q by the system $\{\psi_k\}$.

It remains for us to estimate the error that arises when $C_N(q)$ is replaced by $\bar{C}_N(q)$. In other words, it is required to estimate the value of $\|C_N(q) - \bar{C}_N(q)\|_N = \left(\sum_{j=0}^{N-1} (c_j(q) - \bar{c}_j(q))^2\right)^{\frac{1}{2}}$. To this end we consider the point $C'_N(q) = A_N(C_N(q)) = (c'_0(q), \dots, c'_{N-1}(q))$ and write

$$\|C_N(q) - \bar{C}_N(q)\|_N \leqslant \|C_N(q) - C'_N(q)\|_N + \|C'_N(q) - \bar{C}_N(q)\|_N. \tag{2.19}$$

Further, using the inequality (2.18), we have

$$\begin{aligned} \|C'_N(q) - \bar{C}_N(q)\|_N = \|A_N(C_N(q)) - A_N(\bar{C}_N(q))\|_N \leqslant \\ \kappa_\psi \lambda h \|C_N(q) - \bar{C}_N(q)\|_N. \end{aligned} \tag{2.20}$$

From (2.19) and (2.20) we deduce

$$\|C_N(q) - \bar{C}_N(q)\|_N \leqslant \frac{1}{1 - \kappa_\psi \lambda h} \|C_N(q) - C'_N(q)\|_N. \tag{2.21}$$

To estimate the norm on the right-hand side of the inequality (2.21), we note that, due to the Bessel inequality

$$\|C_N(q) - C'_N(q)\|_N^2 \leqslant \sum_{t=0}^{\infty} (F_{C(q),C_N(q)}(t))^2 \rho(t), \tag{2.22}$$

where

$$\begin{aligned} F_{C(q),C_N(q)}(t) = f\left[t, y(0) + h\sum\nolimits_{j=0}^{\infty} c_j(q)\psi_{1,j+1}(t)\right] \\ - f\left[t, y(0) + h\sum\nolimits_{j=0}^{N-1} c_j(q)\psi_{1,j+1}(t)\right]. \end{aligned}$$

From (2.22) and (2.13) follows that

$$(F_{C(q),C_N(q)}(t))^2 \leqslant \lambda^2 \gamma(t) \left(\sum\nolimits_{j=N}^{\infty} hc_j(q)\psi_{1,j+1}(t)\right)^2,$$

hence taking into account (2.6) we have

$$(F_{C(q),C_N(q)}(t))^2 \leqslant \lambda^2 \gamma(t) \left(\sum\nolimits_{j=N}^{\infty} y_{1,j+1}\psi_{1,j+1}(t)\right)^2. \tag{2.23}$$

Gathering (2.23) with (2.22), we get

$$\|C_N(q) - C'_N(q)\|_N^2 \leqslant \lambda^2 \sum_{t=0}^{\infty} \left(\sum\nolimits_{j=N}^{\infty} y_{1,j+1}\psi_{1,j+1}(t)\right)^2 \rho(t)\gamma(t). \tag{2.24}$$

Summarizing, we can derive the following result from (2.21) and (2.24).

THEOREM 3. *Suppose function $f(x,y)$ is defined and bounded on the Cartesian product $\Omega\times\mathbb{R}$ and satisfies the Lipschitz condition (2.13), h and λ satisfy the inequality $h\lambda\kappa_\psi < 1$, where κ_ψ is defined by the equality (2.3). Let l_2 be the Hilbert space consisting of the sequences $C=(c_0,c_1,\ldots)$, for which norm $\|C\| = \left(\sum_{j=0}^{\infty} c_j^2\right)^{\frac12}$, the operator $A: l_2\to l_2$ which matches a point $C\in l_2$ to point $C'\in l_2$ by the rule (2.9). Moreover, let $A_N:\mathbb{R}^N\to\mathbb{R}^N$ – the finite-dimensional analogue of the operator A which matches a point $C_N=(c_0,\ldots,c_N)\in\mathbb{R}^N$ to point $C'_N=(c'_0,\ldots,c'_N)\in\mathbb{R}^N$ by the rule (2.17). Then operators $A: l_2\to l_2$ and $A_N:\mathbb{R}^N\to\mathbb{R}^N$ are contractions and consequently there are fixed points $C(q)=(c_0(q),c_1(q),\ldots)=A(C(q))\in l_2$ and $\bar C_N(q)=(\bar c_0(q),\bar c_0(q),\ldots,\bar c_N(q)) = A_N(\bar C_N(q))\in\mathbb{R}^N$, for which the following inequality holds*

$$\|C_N(q)-\bar C_N(q)\|_N \leqslant \frac{\lambda\sigma_N^{\psi}(y)}{1-h\kappa_\psi\lambda},$$

where

$$\sigma_N^{\psi}(y) = \left(\sum_{t=0}^{\infty}\left(\sum\nolimits_{j=N+1}^{\infty} y_{1,j}\psi_{1,j}(t)\right)^2\rho(t)\gamma(t)\right)^{\frac12},$$

but $C_N(q)=(c_0(q),\ldots,c_{N-1}(q))$ – is a finite sequence consisting of the first N components of the point $C(q)$, and due to (2.6) also the equality hold $hC_N(q) = (y_{1,1},y_{1,2},\ldots,y_{1,N})$.

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