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Systems of functions orthogonal in the sense of Sobolev associated with Haar functions and the Cauchy problem for ODEs

We consider systems of functions $\mathcal{X}_{r,n}(x)$ ($r = 1, 2, \dots, n = 0, 1, \dots$), generated by Haar functions $\chi_n(x)$ ($n = 1, 2, \dots$), that form the Sobolev orthonormal system with respect to the scalar product of the following form $\langle f, g \rangle = \sum_{\nu=0}^{r-1} f^{(\nu)}(0)g^{(\nu)}(0) + \int_0^1 f^{(r)}(t)g^{(r)}(x)dx$. It is shown that the Fourier series and sums with respect to the system $\mathcal{X}_{r,n}(x)$ ($n = 0, 1, \dots$) are a convenient and very effective tool for the approximate solution of the Cauchy problem for ordinary differential equations (ODEs).

Bibliography: 23 items.

Рассмотрены системы функций $\mathcal{X}_{r,n}(x)$ ($r = 1, 2, \dots, n = 0, 1, \dots$), порожденные функциями Хаара $\chi_n(x)$ ($n = 1, 2, \dots$), образующие ортонормированную по Соболеву систему относительно скалярного произведения следующего вида $\langle f, g \rangle = \sum_{\nu=0}^{r-1} f^{(\nu)}(0)g^{(\nu)}(0) + \int_0^1 f^{(r)}(t)g^{(r)}(x)dx$. Показано, что ряды и суммы Фурье по системе $\mathcal{X}_{r,n}(x)$ ($n = 0, 1, \dots$) является удобным и весьма эффективным инструментом приближенного решения задачи Коши для систем обыкновенных дифференциальных уравнений (ОДУ).

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Introduction

In this paper we continue to consider systems of functions orthogonal with respect to scalar products in which one or more points with discrete masses are present. Interest in such systems has been growing lately (see [1] – [12] and the literature cited there). This new direction is usually denoted by the words: "functions orthogonal on Sobolev". The increased attention of specialists to this direction of the theory of orthogonal systems can also be explained by the fact that the Fourier series on polynomials orthogonal in the sense of Sobolev turned out to be a natural and very convenient tool for representing solutions of differential (difference) equations. This

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can be shown, in particular, by the example of the Cauchy problem for a differential equation

$$F(x, y, y', \dots, y^{(r)}) = 0 \quad (0.1)$$

with initial conditions $y^{(k)}(0) = y_k$, $k = 0, 1, \dots, r-1$. Along with various grid and approximation-analytical methods, so-called spectral methods [13] – [21] are often used to solve this problem. We recall that the essence of the spectral method for solving the Cauchy problem for ODEs (0.1) lies in the fact that, first of all, the required solution $y(x)$ is represented as a Fourier series

$$y(x) = \sum_{k=0}^{\infty} \hat{y}_k \psi_k(x) \quad (0.2)$$

on a suitable orthonormal system $\{\psi_k(x)\}_{k=0}^{\infty}$ (most often as $\{\psi_k(x)\}_{k=0}^{\infty}$ we use a trigonometric system, orthogonal polynomials, wavelets and some others). In the second step in equation (0.1), $y(x)$ is replaced by the series (0.2). This leads to a system of equations for the unknown coefficients $\hat{y}_k$ ($k = 0, 1, \dots$). At the third stage, it is required to solve this system with considering the initial conditions $y^{(k)}(0) = y_k$, $k = 0, 1, \dots, r-1$ the original Cauchy problem. One of the main difficulties that arises at this stage is to choose an orthonormal basis $\{\psi_k(x)\}_{k=0}^{\infty}$, for which the desired solution $y(x)$ of equation (0.1), represented as a series of (0.2), satisfies the initial conditions $y^{(k)}(0) = y_k$, $k = 0, 1, \dots, r-1$. Moreover, since as a result of solving the system of equations for the unknown coefficients $\hat{y}_k$ only a finite number of coefficients with $k = 0, 1, \dots, n$ will be found, it is very important that the partial sum of the series (0.2) of the form $y_n(x) = \sum_{k=0}^n \hat{y}_k \psi_k(x)$, being an approximate solution of the Cauchy problem in question, also satisfies the initial conditions $y_n^{(k)}(0) = y_k$, $k = 0, 1, \dots, r-1$. We show that a basis $\{\psi_k(x) = \mathcal{X}_{r,k}(x)\}_{k=0}^{\infty}$, consisting of functions $\mathcal{X}_{r,k}(x)$, orthonormal on the Sobolev sense with respect to the scalar product

$$\langle f, g \rangle = \sum_{\nu=0}^{r-1} f^{(\nu)}(0)g^{(\nu)}(0) + \int_0^1 f^{(r)}(t)g^{(r)}(x)dx, \quad (0.3)$$

and associated with the Haar functions $\chi_n(x)$ by means of the equalities

$$\mathcal{X}_{r,k}(x) = \frac{x^k}{k!}, \quad k = 0, 1, \dots, r-1, \quad (0.4)$$

$$\mathcal{X}_{r,r+k}(x) = \frac{1}{(r-1)!} \int_0^x (x-t)^{r-1} \chi_{k+1}(t) dt, \quad k = 0, 1, \dots \quad (0.5)$$

has these properties. For this purpose, we note that the Fourier series (0.2) in the case when $\{\psi_k(x) = \mathcal{X}_{r,k}(x)\}_{k=0}^{\infty}$ acquires, as shown below, the following form

$$y(x) = \sum_{k=0}^{r-1} y^{(k)}(0) \frac{x^k}{k!} + \sum_{k=r}^{\infty} \hat{y}_{r,k} \mathcal{X}_{r,k}(x), \quad (0.6)$$

where $\hat{y}_{r,k} = \int_0^1 y^{(r)}(t) \chi_{k-r+1}(t) dt$.

On the other hand, it follows from definition (0.5) that $(\mathcal{X}_{r,k}(0))^{(\nu)} = 0$ for all $0 \leq \nu \leq r-1$, $r \leq k$, therefore, as a function of $y(x)$, represented as a series of (0.6), and the partial sum of this series of the form

$$y_n(x) = \sum_{k=0}^{r-1} y^{(k)}(0) \frac{x^k}{k!} + \sum_{k=r}^n \hat{y}_{r,k} \mathcal{X}_{r,k}(x)$$

satisfy the initial conditions of the Cauchy problem for equation (0.1). This leads to the idea that functions $\mathcal{X}_{r,k}(x)$ orthogonal in the sense of Sobolev relative to the scalar product (0.3), are closely related to the Cauchy problem for equation (0.1). In the future it will be shown (see §5) that this is actually so.

It should be noted that the mixed series of the form (0.6), associated with the Haar functions were the main object of the study of [1] and [2]. From the point of view of the geometry of Hilbert spaces, the general idea underlying the construction of mixed series is as follows. Suppose that the system of functions $\{\varphi_k(x)\}$ is orthonormal on (a, b) with weight $\rho(x)$, i.e.

$$\int_a^b \varphi_k(x) \varphi_l(x) \rho(x) dx = \delta_{kl}, \quad (0.7)$$

where δ_{kl} – is the Kronecker symbol. We denote by $L_\rho^p(a, b)$ the space of functions $f(x)$, that are measurable on (a, b) , for which

$$\int_a^b |f(x)|^p \rho(x) dx < \infty.$$

If $\rho(x) \equiv 1$, then we write $L_\rho^p(a, b) = L^p(a, b)$ и $L(a, b) = L^1(a, b)$. It follows from (0.7) that $\varphi_k(x) \in L_\rho^2(a, b)$ ($k = 0, 1, \dots$). We add to this condition one more, considering that $\varphi_k(x) \in L(a, b)$ ($k = 0, 1, \dots$). Then we can define the following $\{\varphi_k(x)\}$ -generated functions:

$$\varphi_{r,r+k}(x) = \frac{1}{(r-1)!} \int_a^x (x-t)^{r-1} \varphi_k(t) dt, \quad k = 0, 1, \dots \quad (0.8)$$

In addition, we define a finite set of functions

$$\varphi_{r,k}(x) = \frac{(x-a)^k}{k!}, \quad k = 0, 1, \dots, r-1. \quad (0.9)$$

From (0.8) and (0.9) it follows that for a.e. $x \in (a, b)$

$$(\varphi_{r,k}(x))^{(\nu)} = \begin{cases} \varphi_{r-\nu,k-\nu}(x), & \text{if } 0 \leq \nu \leq r-1, r \leq k, \\ \varphi_{k-r}(x), & \text{if } \nu = r \leq k, \\ \varphi_{r-\nu,k-\nu}(x), & \text{if } \nu \leq k < r, \\ 0, & \text{if } k < \nu \leq r-1. \end{cases} \quad (0.10)$$

We denote by $W_{L^p_\rho(a,b)}^r$ the Sobolev space $W_{L^p_\rho(a,b)}^r$, consisting of functions $f(x)$, that are continuously differentiable on $[a, b]$ $r - 1$ times, and $f^{(r-1)}(x)$ is absolutely continuous on $[a, b]$ and $f^{(r)}(x) \in L^p_\rho(a, b)$. The scalar product in the space $W_{L^p_\rho(a,b)}^r$ is defined by the equality

$$\langle f, g \rangle = \sum_{\nu=0}^{r-1} f^{(\nu)}(a)g^{(\nu)}(a) + \int_a^b f^{(r)}(t)g^{(r)}(t)\rho(t)dt. \quad (0.11)$$

Then, using the definition of the functions $\varphi_{r,k}(x)$ (cm. (0.8) and (0.9)) and the equality (0.10), it is easy to see (see Theorem 2.1) that the system $\{\varphi_{r,k}(x)\}_{k=0}^\infty$ is orthonormal in the space $W_{L^p_\rho(a,b)}^r$. We call the system $\{\varphi_{r,k}(x)\}_{k=0}^\infty$ *orthonormal in the sense by Sobolev* of the scalar product of (0.11) and the *generated* orthonormal system $\{\varphi_k(x)\}_{k=0}^\infty$. In what follows it will be shown (see §2), that the Fourier series of the function $f(x) \in W_{L^p_\rho(a,b)}^r$ in the system $\{\varphi_{r,k}(x)\}_{k=0}^\infty$ has a mixed character, namely:

$$f(x) \sim \sum_{k=0}^{r-1} f^{(k)}(a) \frac{(x-a)^k}{k!} + \sum_{k=r}^\infty \hat{f}_{r,k} \varphi_{r,k}(x), \quad (0.12)$$

where

$$\hat{f}_{r,k} = \int_a^b f^{(r)}(t) \varphi_{r,k}^{(r)}(t) \rho(t) dt = \int_a^b f^{(r)}(t) \varphi_{k-r}(t) \rho(t) dt, \quad (0.13)$$

therefore a series of the form (0.12) will be called a *mixed series* according to the system $\{\varphi_k(x)\}_{k=0}^\infty$, considering this name as a conditional and abbreviated designation of the full name: «*the Fourier series in the system $\{\varphi_{r,k}(x)\}_{k=0}^\infty$, orthonormalized by the Sobolev generated by the orthonormal system $\{\varphi_k(x)\}_{k=0}^\infty$* ».

We note some important properties of the mixed series (0.12), which directly follow from (0.10). The first of these is related to the differential property of the mixed series, namely, if $r > 1$, then as a result of termwise differentiation of the mixed series (0.12) we obtain a mixed series for the derivative $f'(x)$, corresponding to the case when Instead of r , $r - 1$ appears, in other words

$$f'(x) \sim \sum_{k=1}^\infty f'_{r-1,k-1} \varphi_{r-1,k-1}(x) = \sum_{k=1}^\infty (\hat{f}_{r,k} \varphi_{r,k}(x))'. \quad (0.14)$$

The second property is related to term-by-term integration with a variable upper limit and has the form

$$\int_a^x f'(t) dt \sim \sum_{k=1}^\infty f'_{r-1,k-1} \int_a^x \varphi_{r-1,k-1}(t) dt = \sum_{k=1}^\infty \hat{f}_{r,k} \varphi_{r,k}(x).$$

Of importance is the property of the mixed series (0.12), which consists in the fact that its partial sum of the form

$$Y_{r,N}(f, x) = \sum_{k=0}^{r-1} f^{(k)}(a) \frac{(x-a)^k}{k!} + \sum_{k=r}^N \hat{f}_{r,k} \varphi_{r,k}(x) \quad (0.15)$$

for $r \leq N$ coincides with the initial function $f(x)$ at the point $x = a$ r -fold, that is

$$(Y_{r,N}(f, x))_{x=a}^{(\nu)} = f^{(\nu)}(a) \quad (0 \leq \nu \leq r-1).$$

In addition, from (0.10) and (0.15) it follows that $(0 \leq \nu \leq r-1)$

$$Y_{r,N}^{(\nu)}(f, x) = \sum_{n=0}^{r-1-\nu} f^{(n+\nu)}(a) \frac{(x-a)^n}{n!} + \sum_{n=r-\nu}^{N-\nu} f_{r-\nu,n}^{(\nu)} \varphi_{r-\nu,n}(x) = Y_{r-\nu,N-\nu}(f^{(\nu)}, x),$$

hence, in turn, we derive in $(0 \leq \nu \leq r-2)$

$$\begin{aligned} f^{(\nu)}(x) - Y_{r,N}^{(\nu)}(f, x) &= \frac{1}{(r-\nu-2)!} \int_a^x (x-t)^{r-\nu-2} (f^{(r-1)}(t) - Y_{r,N}^{(r-1)}(f, t)) dt = \\ &= \frac{1}{(r-\nu-2)!} \int_a^x (x-t)^{r-\nu-2} (f^{(r-1)}(t) - Y_{1,N-r+1}(f^{(r-1)}, t)) dt. \end{aligned}$$

We shall need below some results concerning questions of the convergence of mixed series of the form (0.12).

1. Some general results

We first consider the completeness problem in $W_{L_\rho^2(a,b)}^r$ of the system $\{\varphi_{r,k}(x)\}_{k=0}^\infty$, consisting of functions defined by the equalities (0.8) and (0.9).

THEOREM 1. *Suppose that the functions $\varphi_k(x)$ ($k = 0, 1, \dots$) form a complete in $L_\rho^2(a,b)$ orthonormal system with weight $\rho(x)$ on (a,b) . Then the system $\{\varphi_{r,k}(x)\}_{k=0}^\infty$, generated by the system $\{\varphi_k(x)\}_{k=0}^\infty$ by means of the equalities (0.8) and (0.9), is complete in $W_{L_\rho^2(a,b)}^r$ and orthonormal with respect to the scalar product (0.11).*

PROOF. From the equality (0.10) it follows that if $r \leq k$ and $0 \leq \nu \leq r-1$, then $(\varphi_{r,k}(x))_{x=a}^{(\nu)} = 0$, therefore, by virtue of (0.7), (0.10), we have

$$\begin{aligned} \langle \varphi_{r,k}, \varphi_{r,l} \rangle &= \int_a^b (\varphi_{r,k}(x))^{(r)} (\varphi_{r,l}(x))^{(r)} \rho(x) dx = \\ &= \int_a^b \varphi_{k-r}(x) \varphi_{l-r}(x) \rho(x) dx = \delta_{kl}, \quad k, l \geq r, \end{aligned}$$

and from (0.9) and (0.10) we have

$$\langle \varphi_{r,k}, \varphi_{r,l} \rangle = \sum_{\nu=0}^{r-1} (\varphi_{r,k}(x))^{(\nu)}|_{x=a} (\varphi_{r,l}(x))^{(\nu)}|_{x=a} = \delta_{kl}, \quad k, l < r.$$

It is also obvious that $\langle \varphi_{r,k}, \varphi_{r,l} \rangle = 0$, if $k < r \leq l$ or $l < r \leq k$.

This means, that functions $\varphi_{r,k}(t)$ ($k = 0, 1, \dots$) form an orthonormal system in $W_{L_\rho^2(a,b)}^r$ with respect to the scalar product (0.11). It remains to be convinced of its completeness in $W_{L_\rho^2(a,b)}^r$. To this end, we show that if for some function $f = f(x) \in W_{L_\rho^2(a,b)}^r$ and for all $k = 0, 1, \dots$ the equalities $\langle f, \varphi_{r,k} \rangle = 0$ are valid,

then $f(x) \equiv 0$. In fact, if $k \leq r-1$, then $\langle f, \varphi_{r,k} \rangle = f^{(k)}(a)$, therefore taking into account that $\langle f, \varphi_{r,k} \rangle = 0$, for our function $f(x)$ the Taylor formula takes the form

$$f(x) = \frac{1}{(r-1)!} \int_a^x (x-t)^{r-1} f^{(r)}(t) dt.$$

On the other hand, for all $k \geq r$ we have

$$0 = \langle f, \varphi_{r,k} \rangle = \int_a^b f^{(r)}(x) (\varphi_{r,k}(x))^{(r)} \rho(x) dx = \int_a^b f^{(r)}(x) \varphi_{k-r}(x) \rho(x) dx.$$

From this and from the fact that $\varphi_m(x)$ ($m = 0, 1, \dots$) form a complete orthonormal system in $L_\rho^2(a, b)$ we have $f^{(r)}(x) = 0$ almost everywhere on (a, b) . Therefore $f(x) \equiv 0$. Theorem 1 is proved.

THEOREM 2. Assume that the $\frac{1}{\rho(x)} \in L(a, b)$, and $\varphi_k(x)$ ($k = 0, 1, \dots$) functions form complete in $L_\rho^2(a, b)$ orthonormal with weight $\rho(x)$ on (a, b) , $\{\varphi_{r,k}(x)\}_{k=0}^\infty$ is a system orthonormalized in $W_{L_\rho^2(a,b)}^r$ with respect to the scalar product (0.11), generated by the system $\{\varphi_k(x)\}_{k=0}^\infty$ by means of the equations (0.8) and (0.9). Then, if $f(x) \in W_{L_\rho^2(a,b)}^r$, the Fourier series (the mixed series) (0.12) converges to the function $f(x)$ uniformly with respect to $x \in [a, b]$.

PROOF. We denote by $S_n(f^{(r)}) = S_n(f^{(r)}, x)$ the partial sum of the Fourier series of the function $f^{(r)}(x) \in L_\rho^2(a, b)$ by the system $\{\varphi_k(x)\}_{k=0}^n$, i.e.

$$S_n(f^{(r)}, x) = \sum_{k=0}^n f_{r,k+r} \varphi_k(x),$$

where the coefficients $f_{r,k+r}$ ($k = 0, 1, \dots$) are defined by (0.13). It follows from the conditions of Theorem 2 that for $n \rightarrow \infty$

$$\|f^{(r)} - S_n(f^{(r)})\|_{L_\rho^2(a,b)} \rightarrow 0. \quad (1.1)$$

Further, let $Y_{n+r}(f, x)$ denote the partial sum of the mixed series (0.12) of the following form

$$Y_{n+r}(f, x) = \sum_{k=0}^{r-1} f^{(k)}(a) \frac{(x-a)^k}{k!} + \sum_{k=r}^{n+r} \hat{f}_{r,k} \varphi_{r,k}(x), \quad (1.2)$$

and write the Taylor formula

$$f(x) = \sum_{k=0}^{r-1} f^{(k)}(a) \frac{(x-a)^k}{k!} + \frac{1}{(r-1)!} \int_a^x (x-t)^{r-1} f^{(r)}(t) dt. \quad (1.3)$$

From (1.2) and (1.3) we have

$$f(x) - Y_{n+r}(f, x) = \frac{1}{(r-1)!} \int_a^x (x-t)^{r-1} f^{(r)}(t) dt - \sum_{k=r}^{n+r} \hat{f}_{r,k} \varphi_{r,k}(x). \quad (1.4)$$

Turning to the equality (0.8), then (1.4) can be rewritten as

$$\begin{aligned} f(x) - Y_{n+r}(f, x) = \\ \frac{1}{(r-1)!} \int_a^x (x-t)^{r-1} f^{(r)}(t) dt - \frac{1}{(r-1)!} \int_a^x (x-t)^{r-1} \sum_{k=r}^{n+r} \hat{f}_{r,k} \varphi_{k-r}(t) dt = \\ \frac{1}{(r-1)!} \int_a^x (x-t)^{r-1} [f^{(r)}(t) - S_n(f^{(r)}, t)] dt. \end{aligned} \quad (1.5)$$

From (1.5) and the Holder inequality we have

$$|f(x) - Y_{n+r}(f, x)| \leq \frac{1}{(r-1)!} \left(\int_a^b \frac{|x-t|^{2(r-1)}}{\rho(t)} dt \right)^{\frac{1}{2}} \left(\int_a^b [f^{(r)}(t) - S_n(f^{(r)}, t)]^2 \rho(t) dt \right)^{\frac{1}{2}}. \quad (1.6)$$

Comparing (1.1) with (1.6), we see that Theorem 2 holds.

2. Some information about the Haar system

When we study the system of functions $\mathcal{X}_{r,n}(x)$ ($n = 0, 1, \dots$), orthogonal in the sense of Sobolev generated by the Haar $\varphi_k(x) = \chi_{k+1}(x)$ ($k = 0, 1, \dots$) functions, we need the following notation [22]. Suppose that $n = 2^k + i$, $i = 1, 2, \dots, 2^k$, $k = 0, 1, \dots$, $\Delta_n = \Delta_k^i = (\frac{i-1}{2^k}, \frac{i}{2^k})$, $\bar{\Delta}_n = [\frac{i-1}{2^k}, \frac{i}{2^k}]$, ($n \geq 2$), $\Delta_1 = [0, 1]$,

$$\Delta_n^+ = (\Delta_k^i)^+ = (\frac{i-1}{2^k}, \frac{2i-1}{2^{k+1}}) = \Delta_{k+1}^{2i-1}, \quad \Delta_n^- = (\Delta_k^i)^- = (\frac{2i-1}{2^{k+1}}, \frac{i}{2^k}) = \Delta_{k+1}^{2i}.$$

The Haar system is a system of functions, $\chi = \{\chi_n\}_{n=1}^\infty(x)$, $x \in [0, 1]$, in which $\chi_1(x) = 1$, and the function $\chi_n(x)$ with $2^k < n \leq 2^{k+1}$, $k = 0, 1, \dots$ is defined as

$$\chi_n(x) = \begin{cases} 0, & \text{if } x \notin \bar{\Delta}_n, \\ 2^{k/2}, & \text{if } x \in \Delta_n^+, \\ -2^{k/2}, & \text{if } x \in \Delta_n^-. \end{cases} \quad (2.1)$$

The values at the break points of the function $\chi_n(x)$ are chosen in such a way that the following equalities hold:

$$\begin{aligned} \chi_n(x) &= \lim_{\delta \rightarrow 0} \frac{1}{2} [\chi_n(x+\delta) + \chi_n(x-\delta)], \quad x \in (0, 1), \\ \chi_n(0) &= \lim_{\delta \rightarrow 0} \chi_n(\delta), \quad \chi_n(1) = \lim_{\delta \rightarrow 0} \chi_n(1-\delta). \end{aligned}$$

Directly from (2.1) we derive the following orthogonality property of Haar functions:

$$\int_0^1 \chi_n(x) \chi_m(x) dx = \delta_{nm}.$$

If the function $f = f(x)$ is integrable on $[0, 1]$, then we can define the Fourier-Haar coefficients

$$\hat{f}_{0,k} = \int_0^1 f(t) \chi_k(t) dt$$

and the Fourier-Haar series

$$f(x) = \sum_{k=1}^{\infty} \hat{f}_{0,k} \chi_k(x).$$

We denote by $Q_n(f, x)$ the partial Fourier-Haar sum of the form

$$Q_n(f) = Q_n(f, x) = \sum_{k=1}^n \hat{f}_{0,k} \chi_k(x).$$

It is well known [22], that if $p \geq 1$ and $f \in L^p(0, 1)$, then

$$\int_0^1 |f(x) - Q_n(f, x)|^p dx \rightarrow 0 \quad (n \rightarrow \infty).$$

3. Sobolev orthonormal system of functions generated by Haar functions

We set $\varphi_k(x) = \chi_{k+1}(x)$ ($k = 0, 1, \dots$), and for every natural r we define a system of functions $\{\mathcal{X}_{r,n}\}_{n=r}^{\infty}$ as follows:

$$\mathcal{X}_{r,k+r}(x) = \frac{1}{(r-1)!} \int_0^x (x-t)^{r-1} \varphi_k(x) dt, \quad k = 0, 1, 2, \dots \quad (3.1)$$

In addition, we define a finite set of functions

$$\mathcal{X}_{r,n}(x) = \frac{x^n}{n!}, \quad n = 0, 1, \dots, r-1. \quad (3.2)$$

Let $W_{L^2(0,1)}^r$ be the Sobolev space with $L^2(0, 1) = L_{\rho}^2(0, 1)$, and $\rho = \rho(x) \equiv 1$. It follows directly from Theorem 1

COROLLARY 1. *The system $\{\mathcal{X}_{r,n}(x)\}_{n=0}^{\infty}$, generated by the Haar system $\{\chi_n(x)\}_{n=1}^{\infty}$ by means of the equalities (3.1) and (3.2), is complete in $W_{L^2(0,1)}^r$ and orthonormal relative to the scalar product*

$$\langle f, g \rangle = \sum_{\nu=0}^{r-1} f^{(\nu)}(0) g^{(\nu)}(0) + \int_0^1 f^{(r)}(t) g^{(r)}(t) dt. \quad (3.3)$$

The Fourier series by the system $\{\mathcal{X}_{r,n}(x)\}_{n=0}^{\infty}$ has the following form (see (0.6))

$$f(x) \sim \sum_{n=0}^{r-1} f^{(n)}(0) \frac{x^n}{n!} + \sum_{n=r}^{\infty} \hat{f}_{r,n} \mathcal{X}_{r,n}(x), \quad (3.4)$$

where

$$\hat{f}_{r,n} = \int_0^1 f^{(r)}(t) \varphi_{n-r}(t) dt = \int_0^1 f^{(r)}(t) \chi_{n-r+1}(t) dt.$$

Using the notation different from the one adopted in this paper, the following representation is obtained in [1] for functions $\mathcal{X}_{r,n}(x)$, defined by the equality (3.1) ($n = 2^k + i$, $i = 1, 2, \dots, 2^k$, $k = 0, 1, \dots$)

$$\mathcal{X}_{r,n+r-1}(x) = \frac{2^{\frac{k}{2}}}{r!} \begin{cases} 0, & 0 \leq x \leq \frac{i-1}{2^k}; \\ (x - \frac{i-1}{2^k})^r, & \frac{i-1}{2^k} \leq x \leq \frac{2i-1}{2^{k+1}}; \\ (x - \frac{i-1}{2^k})^r - 2(x - \frac{2i-1}{2^{k+1}})^r, & \frac{2i-1}{2^{k+1}} \leq x \leq \frac{i}{2^k}; \\ (x - \frac{i-1}{2^k})^r - 2(x - \frac{2i-1}{2^{k+1}})^r + (x - \frac{i}{2^k})^r, & \frac{i}{2^k} \leq x \leq 1 \end{cases} \quad (3.5)$$

and, as a consequence, it is proved that

$$\mathcal{X}_{r,n+r-1}(x) = \frac{2^{k/2}}{r!} \Delta_{\frac{1}{2^{k+1}}}^2 (x-t)_+^r \Big|_{t=\frac{i-1}{2^k}}, \quad (3.6)$$

where $\Delta_h^2 g(t)$ is a finite difference of the second order with step h , $a_+^r = a^r$, if $a \geq 0$ and $a_+^r = 0$, if $a < 0$. We note a special case of the equality (3.5) with $r = 1$:

$$\mathcal{X}_{1,n}(x) = 2^{\frac{k}{2}} \begin{cases} 0, & 0 \leq x \leq \frac{i-1}{2^k}; \\ x - \frac{i-1}{2^k}, & \frac{i-1}{2^k} \leq x \leq \frac{2i-1}{2^{k+1}}; \\ \frac{i}{2^k} - x, & \frac{2i-1}{2^{k+1}} \leq x \leq \frac{i}{2^k}; \\ 0, & \frac{i}{2^k} \leq x \leq 1. \end{cases} \quad (3.7)$$

The following inequality follows from (3.6):

$$|\mathcal{X}_{1,n}(x)| \leq 2^{-\frac{k}{2}-1}, \quad 0 \leq x \leq 1, n = 2^k + i, \quad i = 1, \dots, 2^k, k = 0, \dots \quad (3.8)$$

We also note that the system of functions $F_0(x) = \mathcal{X}_{1,0}(x) = 1$, $F_1(x) = \mathcal{X}_{1,1}(x) = x$, $F_n(x) = 2 \max_{0 \leq t \leq 1} |\chi_n(t)| \mathcal{X}_{1,n}(x)$, $n = 2, \dots$, introduced for the first time in [23] is a classical Faber-Schauder system that is a Schauder basis in the space of continuous functions $C[0,1]$. However, as far as we know, the Faber-Schauder system was not previously considered as Sobolev orthogonal with respect to the scalar product of (3.3) and was not used in this capacity.

In [1] and [2] using the properties (3.5) and (3.6), the approximation properties of the partial sums of the series (3.4) of the following form

$$Y_{r,N}(f, x) = \sum_{n=0}^{r-1} f^{(n)}(0) \frac{x^n}{n!} + \sum_{n=r}^N \hat{f}_{r,n} \mathcal{X}_{r,n}(x)$$

are investigated for functions f from classes $W_{L^p(0,1)}^r$. In particular, the following result is obtained in [1], [2] (see, for example, Theorem 3 of [2]). Let $r \geq 1$, $f \in W_{L(0,1)}^r$, $n = N - r + 1$. Then the following estimates hold

$$|f^{(r-1)}(x) - Y_{r,N}^{(r-1)}(f, x)| \leq \omega_2 \left(f^{(r-1)}, \frac{1}{n} \right), \quad x \in [0, 1],$$

$$|f^{(\nu)}(x) - Y_{r,N}^{(\nu)}(f, x)| \leq \frac{x^{r-\nu-1}}{(r-\nu-1)!} \omega_2 \left(f^{(r-1)}, \frac{1}{n} \right), \quad x \in [0, 1], \quad 0 \leq \nu \leq r-2,$$

where $\omega_2(g, \delta) = \sup_{\substack{0 \leq h \leq \delta, \\ h \leq x \leq 1-h}} |g(x+h) + g(x-h) - 2g(x)|$ is the modulus of continuity of the second order for the function g .

4. On the representation of the solution of the Cauchy problem for ODEs by the Fourier series with respect to the functions $\mathcal{X}_{r,n}(x)$

For simplicity of the computations, we restrict ourselves to the Cauchy problem of the form

$$y'(x) = f(x, y), \quad y(a) = y_0, \quad (4.1)$$

in which the function $f(x, y)$ is assumed to be continuous in some domain G of variables (x, y) , for which the point (a, y_0) is internal. It is required to approximate with a given accuracy the function $y = y(x)$ defined on $[a, b]$, which is a solution of the Cauchy problem (4.1). We divide the interval $[a, b]$ into m parts with step h and let $[a_i, a_{i+1}]$ be the i -th fission interval, in particular, $[a_0, a_1] = [a, a+h]$. We approximate the required solution on the segments $[a_i, a_{i+1}]$, $0 \leq i \leq m-1$, stopping for definiteness on $i = 0$. Setting $x = a + ht$, we map the linear interval $[0, 1]$ onto $[a, a+h]$ linearly. Relatively the new variable $t \in [0, 1]$ the equation (4.1) takes the following form

$$\varphi'(t) = hf(a + ht, \varphi(t)), \quad \varphi(0) = y_0, \quad 0 \leq t \leq 1, \quad (4.2)$$

where $\varphi(t) = y(a + ht)$. Since by assumption the function $f(x, y)$ is continuous in the domain G , it follows from (4.2) that the function $\varphi'(t)$ is continuous on $[0, 1]$ and hence $\varphi \in W_{L^2(0,1)}^1$, therefore, by Theorem 2, we can represent function φ in the form of a Fourier series uniformly convergent on $[0, 1]$ (3.4) with $r = 1$:

$$\varphi(t) = \varphi(0) + \sum_{k=1}^{\infty} \hat{\varphi}_{1,k} \mathcal{X}_{1,k}(t), \quad (4.3)$$

where

$$\hat{\varphi}_{1,k} = \int_0^1 \varphi'(t) \chi_k(t) dt \quad (k \geq 1). \quad (4.4)$$

Our goal is to construct an iterative process to find the approximate values of the coefficients $c_k = \hat{\varphi}_{1,k}/h$ ($k = 1, 2, \dots$). For this, we turn to the relations (3.1), (0.10) and (0.14), which together with (4.3) give

$$\varphi'(t) = \sum_{k=0}^{\infty} \hat{\varphi}_{1,k} \chi_k(t).$$

We set $\phi(t) = f(a + ht, \varphi(t))$ and note that, in view of (4.2) and (4.4) the Fourier-Haar coefficients of the function $\phi = \phi(t)$ have the form

$$c_k(\phi) = \int_0^1 \phi(t) \chi_k(t) dt = \frac{1}{h} \hat{\varphi}_{1,k} \quad (k \geq 1). \quad (4.5)$$

With these equalities, we can rewrite (4.3) in the following form:

$$\varphi(t) = \varphi(0) + h \sum_{k=1}^{\infty} c_k(\phi) \mathcal{X}_{1,k}(t). \quad (4.6)$$

From (4.5), (4.6), in turn, we derive the following relations

$$c_k(\phi) = \int_0^1 f \left[a + ht, \varphi(0) + h \sum_{j=1}^{\infty} c_j(\phi) \mathcal{X}_{1,j}(t) \right] \chi_k(t) dt, \quad k = 1, 2, \dots \quad (4.7)$$

We introduce the Hilbert space l_2 , consisting of sequences $C = (c_1, \dots)$, for which the norm $\|C\| = \left(\sum_{j=1}^{\infty} c_j^2 \right)^{\frac{1}{2}}$ is introduced. In the space l_2 we consider the operator A , which maps the point $C \in l_2$ to the point $C' \in l_2$ by the rule

$$c'_k = \int_0^1 f \left[a + ht, \varphi(0) + h \sum_{j=1}^{\infty} c_j \mathcal{X}_{1,j}(t) \right] \chi_k(t) dt, \quad k = 1, 2, \dots \quad (4.8)$$

It follows from (4.7) that the point $C(\phi) = (c_1(\phi), \dots)$ is a fixed point of the operator $A : l_2 \rightarrow l_2$. In order to find the point $C(\phi)$ by the simple iteration method, it suffices to show that the operator $A : l_2 \rightarrow l_2$ is contractive in the metric of the space l_2 . For this purpose we consider two points $P, Q \in l_2$, where $P = (p_1, \dots)$, $Q = (q_1, \dots)$ and put $P' = A(P)$, $Q' = A(Q)$. We have

$$p'_k - q'_k = \int_0^1 F_{P,Q}(t) \chi_k(t) dt, \quad k = 1, \dots \quad (4.9)$$

where

$$F_{P,Q}(t) = f \left[a + ht, \varphi(0) + h \sum_{j=1}^{\infty} p_j \mathcal{X}_{1,j}(t) \right] - f \left[a + ht, \varphi(0) + h \sum_{j=1}^{\infty} q_j \mathcal{X}_{1,j}(t) \right]. \quad (4.10)$$

From (4.9), using the Bessel inequality, we find

$$\sum_{k=1}^{\infty} (p'_k - q'_k)^2 \leq \int_0^1 (F_{P,Q}(t))^2 dt. \quad (4.11)$$

Suppose that the function $f(x, y)$ satisfies the Lipschitz condition

$$|f(x, y') - f(x, y'')| \leq \lambda |y' - y''|, \quad a \leq x \leq a + h \quad (4.12)$$

in the variable y . From (4.10) and (4.12) we have

$$(F_{P,Q}(t))^2 \leq (h\lambda)^2 \left(\sum_{j=1}^{\infty} (p_j - q_j) \mathcal{X}_{1,j}(t) \right)^2, \quad (4.13)$$

from which, using the Cauchy-Bunyakovskii inequality, we derive

$$(F_{P,Q}(t))^2 \leq (h\lambda)^2 \sum_{j=1}^{\infty} (p_j - q_j)^2 \sum_{j=1}^{\infty} (\mathcal{X}_{1,j}(t))^2$$

Comparing (4.13) with (4.11), we find

$$\sum_{k=1}^{\infty} (p'_k - q'_k)^2 \leq (h\lambda)^2 \sum_{k=1}^{\infty} (p_k - q_k)^2 \int_0^1 \sum_{j=1}^{\infty} (\mathcal{X}_{1,j}(t))^2 dt. \quad (4.14)$$

It remains to estimate the last integral. To do this, we turn to the equality (3.7) and the inequality (3.8), from which we note that the series appearing under the integral in (4.14), converges uniformly for $t \in [0, 1]$. Put

$$\kappa = \left(\int_0^1 \sum_{j=1}^{\infty} (\mathcal{X}_{1,j}(t))^2 dt \right)^{\frac{1}{2}}. \quad (4.15)$$

Then from (4.14) and (4.15) we finally find

$$\left(\sum_{k=1}^{\infty} (p'_k - q'_k)^2 \right)^{\frac{1}{2}} \leq h\kappa\lambda \left(\sum_{k=1}^{\infty} (p_k - q_k)^2 \right)^{\frac{1}{2}}. \quad (4.16)$$

The inequality (4.16) shows that if $h\kappa\lambda < 1$, then the operator $A : l_2 \rightarrow l_2$ is contractive and, as a consequence, the iterative process $C^{\nu+1} = A(C^{\nu})$ converges to the point $C(\phi)$ for $\nu \rightarrow \infty$. However, from the point of view of applications it is important to consider the finite-dimensional analogue of the operator A . We consider the operator $A_N : \mathbb{R}^N \rightarrow \mathbb{R}^N$, that corresponds to the point $C_N = (c_1, \dots, c_N) \in \mathbb{R}^N$ by the point $C'_N = (c'_1, \dots, c'_N) \in \mathbb{R}^N$ by the rule

$$c'_k = \int_0^1 f \left[a + ht, \varphi(0) + h \sum_{j=1}^N c_j \mathcal{X}_{1,j}(t) \right] \chi_k(t) dt, \quad k = 1, 2, \dots, N. \quad (4.17)$$

Consider two points $P_N, Q_N \in \mathbb{R}^N$, where $P_N = (p_1, p_2, \dots, p_N)$, $Q_N = (q_1, q_2, \dots, p_N)$ and set $P'_N = A_N(P_N)$, $Q'_N = A_N(Q_N)$. Repeating the arguments that led us to the inequality (4.16) verbatim, we get

$$\left(\sum_{k=1}^N (p'_k - q'_k)^2 \right)^{\frac{1}{2}} \leq h\kappa\lambda \left(\sum_{k=1}^N (p_k - q_k)^2 \right)^{\frac{1}{2}}. \quad (4.18)$$

The inequality (4.18) shows that if $h\kappa\lambda < 1$, then the $A_N : \mathbb{R}^N \rightarrow \mathbb{R}^N$ operator is contractive and, as a consequence, the iterative process $C_N^{\nu+1} = A_N(C_N^{\nu})$ for $\nu \rightarrow \infty$ converges to its fixed point, which we denote by $\bar{C}_N(\phi) = (\bar{c}_1(\phi), \dots, \bar{c}_N(\phi))$. On the other hand, consider the point $C_N(\phi) = (c_1(\phi), \dots, c_N(\phi))$, composed of the desired Fourier-Haar coefficients of the function ϕ . It remains for us to estimate the error resulting from the replacement of the point $C_N(\phi)$ by the $\bar{C}_N(\phi)$ point. In other words, it is required to estimate the $\|C_N(\phi) - \bar{C}_N(\phi)\|_N = \left(\sum_{j=1}^N (c_j(\phi) - \bar{c}_j(\phi))^2 \right)^{\frac{1}{2}}$ value. To this end, consider the point $C'_N(\phi) = A_N(C_N(\phi)) = (c'_1(\phi), \dots, c'_N(\phi))$ and write

$$\|C_N(\phi) - \bar{C}_N(\phi)\|_N \leq \|C_N(\phi) - C'_N(\phi)\|_N + \|C'_N(\phi) - \bar{C}_N(\phi)\|_N. \quad (4.19)$$

Further, using the inequality (4.18), we have

$$\begin{aligned} \|C'_N(\phi) - \bar{C}_N(\phi)\|_N &= \|A_N(C_N(\phi)) - A_N(\bar{C}_N(\phi))\| \leq \\ &h\kappa\lambda \|C_N(\phi) - \bar{C}_N(\phi)\|_N. \end{aligned} \quad (4.20)$$

From (4.19) and (4.20) we derive

$$\|C_N(\phi) - \bar{C}_N(\phi)\|_N \leq \frac{1}{1 - h\kappa\lambda} \|C_N(\phi) - C'_N(\phi)\|_N. \quad (4.21)$$

To estimate the norm on the right-hand side of the inequality (4.21), note that, by the Bessel inequality

$$\|C_N(\phi) - C'_N(\phi)\|_N^2 \leq \int_0^1 (F_{C(\phi), C_N(\phi)}(t))^2 dt, \quad (4.22)$$

where

$$\begin{aligned} F_{C(\phi), C_N(\phi)}(t) = & f \left[a + ht, \varphi(0) + h \sum_{j=1}^{\infty} c_j(\phi) \mathcal{X}_{1,j}(t) \right] \\ & - f \left[a + ht, \varphi(0) + h \sum_{j=1}^N c_j(\phi) \mathcal{X}_{1,j}(t) \right]. \end{aligned} \quad (4.23)$$

From (4.23) and (4.13) it follows that

$$(F_{C(\phi), C_N(\phi)}(t))^2 \leq \lambda^2 \left(\sum_{j=N+1}^{\infty} h c_j(\phi) \mathcal{X}_{1,j}(t) \right)^2,$$

hence considering (4.5) we have

$$(F_{C(\phi), C_N(\phi)}(t))^2 \leq \lambda^2 \left(\sum_{j=N+1}^{\infty} \hat{\varphi}_{1,j} \mathcal{X}_{1,j}(t) \right)^2. \quad (4.24)$$

Comparing (4.24) with (4.22), we get

$$\|C_N(\phi) - C'_N(\phi)\|_N^2 \leq \lambda^2 \int_0^1 \left(\sum_{j=N+1}^{\infty} \hat{\varphi}_{1,j} \mathcal{X}_{1,j}(t) \right)^2 dt, \quad (4.25)$$

where according to (4.5)

$$\hat{\varphi}_{1,j} = \int_0^1 \varphi'(t) \chi_j(t) dt \quad (j = 1, 2, \dots)$$

are the Fourier-Haar coefficients of the function $\varphi'(t) = hf(a + ht, \varphi(t))$.

Summing up, we obtain the following result from (4.21) and (4.25).

THEOREM 3. *Suppose that $[a, a + h] \times \mathbb{R} \subset G$, function $f(x, y)$ is continuous in the domain G and satisfies the Lipschitz condition (4.12), and h and λ satisfies the inequality $h\lambda\kappa < 1$, where κ is defined by the equality (4.15). Further, let l_2 be the Hilbert space consisting of sequences $C = (c_1, \dots)$, for which the norm $\|C\| = \left(\sum_{j=1}^{\infty} c_j^2 \right)^{\frac{1}{2}}$ is introduced, the operator $A : l_2 \rightarrow l_2$ associates to point $C \in l_2$ the point $C' \in l_2$ by the rule (4.8). In addition, let $A_N : \mathbb{R}^N \rightarrow \mathbb{R}^N$ – be a finite-dimensional analogue of the operator A , which equals point $C_N = (c_1, \dots, c_N) \in \mathbb{R}^N$ the point $C'_N = (c'_1, \dots, c'_N) \in \mathbb{R}^N$ by the rule (4.17). Then the operators $A : l_2 \rightarrow l_2$ and $A_N : \mathbb{R}^N \rightarrow \mathbb{R}^N$ are contractive and, consequently, there are fixed points of $C(\phi) = (c_0(\phi), c_1(\phi), \dots) = A(C(\phi)) \in l_2$ u $\bar{C}_N(\phi) = (\bar{c}_0(\phi), \bar{c}_1(\phi), \dots, \bar{c}_N(\phi)) = A_N(\bar{C}_N(\phi)) \in \mathbb{R}^N$, for which*

$$\|C_N(\phi) - \bar{C}_N(\phi)\|_N \leq \frac{\lambda}{1 - h\kappa\lambda} \left(\int_0^1 \left(\sum_{j=N+1}^{\infty} \hat{\varphi}_{1,j} \mathcal{X}_{1,j}(t) \right)^2 dt \right)^{\frac{1}{2}} \quad (4.26)$$

holds, where $C_N(\phi) = (c_1(\phi), \dots, c_N(\phi))$ – is a finite sequence composed of the first N components of the point $C(\phi)$, for which, by (4.5) the equality $hC_N(\phi) = (\hat{\varphi}_{1,1}, \hat{\varphi}_{1,2}, \dots, \hat{\varphi}_{1,N})$ also holds.

REMARK 1. The results obtained in Theorem 3 can easily be transferred to systems of differential equations of the form

$$y^{(r)}(x) = f(x, y, y', \dots, y^{(r-1)}), \quad y(a) = y_0,$$

where $f = (f_1, \dots, f_m)$, $y = (y_1, \dots, y_m)$.

We note that the value of $V_N(t) = V_N(\varphi, t) = \sum_{j=N+1}^{\infty} \hat{\varphi}_{1,j} \mathcal{X}_{1,j}(t)$, on the right-hand side of the inequality (4.26), is the remainder term of the Fourier series of the function φ with respect to the functions $\mathcal{X}_{1,j}$ ($k = 0, 1, \dots$), that form the Sobolev orthonormal system with respect to the scalar product (0.3), associated with the Haar functions χ_k by means of the equalities (0.4) and (0.5) with $r = 1$ (as noted above, $\mathcal{X}_{1,j}$ ($k = 0, 1, \dots$) is a Faber-Schauder function up to normalizing factors). The inequality (4.26) directly leads to the problem of investigating the approximate properties of partial sums of the Fourier series (4.3) and more general series (3.4) with arbitrary $r \in \mathbb{N}$ for smooth functions φ . In this paper, we will not dwell on a detailed discussion of this problem, which was the subject of a study of [1] and [2]. A particular case of this problem, related to $r = 1$, has been fairly well studied [22], since it deals with the problem of estimating the value of $V_N(t) = V_N(\varphi, t)$, which is the deviation from the $\varphi(t)$ of the partial sum of the $\sigma_N(\varphi, t)$ series, which is the expansion of the function φ by the Faber-Schauder functions.

References

- [1] Sharapudinov I.I., Muratova G.N. Nekotore svoistva r -kratno integrirovannih ryadov po sisteme Haara // Izv. Sarat. Un-ta Nov. Ser., Ser. Matematika. Mekhanika. Informatika. 2009. Vol. 9. Issue 1. P. 68–76. (in Russian)
- [2] Sharapudinov I.I. Sistemy funktsiy, ortogonalnih po Sobolevu, porozhdenie ortogonalnimi funktsiyami // Izv. RAN. Ser. Matematicheskaya. 2018. (Priniyata k pechati) (in Russian)
- [3] Iserles A., Koch P.E., Norsett S.P., Sanz-Serna J.M. On polynomials orthogonal with respect to certain Sobolev inner products. // J. Approx. Theory. 1991. Vol. 65. Pp. 151–175.
- [4] Marcellan F., Alfaro M. and Rezola M.L. Orthogonal polynomials on Sobolev spaces: old and new directions // Journal of Computational and Applied Mathematics. 1993. Vol. 48. Pp. 113 – 131. North-Holland.
- [5] Meijer H.G. Laguerre polynomials generalized to a certain Laguerre polynomials generalized to a certain discrete Sobolev inner product space // J. Approx. Theory. 1993. Vol. 73. Pp. 1–16.
- [6] Kwon K.H. and Littlejohn L.L. The orthogonality of the Laguerre polynomials $\{L_n^{(-k)}(x)\}$ for positive integers k // Ann. Numer. Anal. Issue 2. Pp. 289–303. 1995.
- [7] Lopez G. Marcellan F. Vanassche W. Relative Asymptotics for Polynomials Orthogonal with Respect to a Discrete Sobolev Inner-Product // Constr. Approx. 1995. Vol. 11(1). Pp. 107–137.
- [8] Kwon K.H. and Littlejohn L.L. Sobolev orthogonal polynomials and second-order differential equations // Rocky Mountain J. Math. 1998. Vol. 28. Pp. 547–594.
- [9] Marcellan F., Yuan Xu. On Sobolev orthogonal polynomials // Expositiones Mathematicae. 2015. Vol. 33. Issue 3. Pp. 308–352.
- [10] Sharapudinov I.I. Mixed series in ultraspherical polynomials and their approximation properties // Sbornik: Mathematics, 2003. Vol. 194. Issue 3. Pp. 423–456.

- [11] Sharapudinov I.I. Smeshannie ryadi po ortogonalnim polimnam. Izdatelstvo Dagestanskogo nauchnogo centra. Makhachakala. 2004. 176 p. (in Russian)
- [12] Sharapudinov I.I. Sistemi funktsii, ortogonalnie po Sobolevu, porojdennie ortogonalnymi funktsiami. // Materialy 18-oi mejdunarodnoi Saratovskoi zimney shkoli «Sovremennye problemy teorii funktsiy i ih prilozheniya». Saratov. OOO «Izdatelstvo «Nauchnaya kniga». 2016. Pp. 329–332. (in Russian)
- [13] Trefethen L.N. Spectral methods in Matlab. SIAM. Philadelphia. 2000.
- [14] Trefethen L.N. Finite difference and spectral methods for ordinary and partial differential equation. Cornell University. 1996.
- [15] Solodovnikov V.V., Dmitriev A.N., Egupov N.D. Spektralnye metody rascheta i proektirovaniya sistem upravleniya. Moskva. Mashinostroyeniye. 1986. (in Russian)
- [16] Pashkovskiy S. Vichislitelnye primeneniya mnogochlenov i ryadov Chebisheva. Moskva. Nauka. 1983. (in Russian)
- [17] Arushunian O.B., Volchenskova N.I., Zaletkin S.F. Primeneniye ryadov Chebisheva dlya integrirovaniya obiknovennykh differentsialnykh uravneniy // Sib. elektron. matem. izv. 2014. Vol. 11. Pp. 517–531. (in Russian)
- [18] Lukomskiy D.S., Terehin P.A. Primeneniye sistem Haara k chislenному resheniyu zadachi Koshi dlya lineinogo differentsialnogo uravneniya pervogo poryadka // Materialy 18-oi mejdunarodnoi Saratovskoi zimney shkoli «Sovremennye problemy teorii funktsiy i ih prilozheniya». Saratov. OOO «Izdatelstvo «Nauchnaya kniga». 2016. Pp. 171–173. (in Russian)
- [19] Magomed-Kasumov M.G. Priblizhennoye resheniye obyknovennykh differentsialnykh uravneniy s ispolzovaniyem smeshannykh ryadov po sisteme Haara // Materialy 18-oi mejdunarodnoi Saratovskoi zimney shkoli «Sovremennye problemy teorii funktsiy i ih prilozheniya». Saratov. OOO «Izdatelstvo «Nauchnaya kniga». 2016. Pp. 176–178. (in Russian)
- [20] Sharapudinov I.I., Magomed-Kasumov M.G. O predstavlenii resheniya zadachi Koshi ryadom Furie po polinomam, ortogonalnim po Sobolevu, porojdennim mnogochlenami Lagerra // Differentsialnye uravneniya. 2017. (prinayta k pechaty) (in Russian)
- [21] Sharapudinov I.I., Sharapudinov T.I. Mixed series of Jacobi and Chebyshev polynomials and their discretization // Mathematical Notes. 2010. Vol. 88. Issue 1. Pp. 112–139.
- [22] Kashin B.S., Saakyan A.A. Ortogonalnye ryadi. AFC. Moskva. 1999. (in Russian)
- [23] Faber G. Ober die Orthogonalfunktionen des Herrn Haar // Jahresber. Deutsch. Math. Verein. 1910. Vol. 19. Pp. 104–112.

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