УДК 517.956.4

I. I. Sharapudinov, T. I. Sharapudinov

Polynomials, orthogonal on Sobolev, derived by the Chebyshev polynomials, orthogonal on the uniform net

In this article we consider the problem of constructing polynomials, orthogonal in Sobolev sense on the finite uniform net and associated with classical Chebyshev polynomials of discrete variable. We have found an explicit expression of these polynomials by classical Chebyshev polynomials. Also we have obtained an expansion of new polynomials by generalized powers of Newton type. We obtain expressions for the deviation of a discrete function and its finite differences from respectively partial sums of its Fourier series on the new system of polynomials and their finite differences.

Bibliography: 30 items.

Рассмотрена задача о конструировании полиномов, ортогональных по Соболеву на конечной равномерной сетке и ассоциированных с классическими полиномами Чебышева дискретной переменной. Установлено явное выражение этих полиномов через классические многочлены Чебышева, получено их разложение по обобщенным степеням ньютоновского типа. Получены выражения для отклонения дискретной функции и ее конечных разностей от, соответственно, ее частичных сумм Фурье по построенной системе полиномов, ортогональных по Соболеву, и их конечных разностей.

Библиография: 30 названий.

Keywords: Polynomials orthogonal in Sobolev sense, Chebyshev polynomials orthogonal on the grid, approximation of discrete functions, mixed series of Chebyshev polynomials orthogonal on a uniform grid.

Ключевые слова: полиномы, ортогональные по Соболеву; полиномы Чебышева, ортогональные на сетке; приближение дискретных функций, смешанные ряды по полиномам Чебышева, ортогональным на равномерной сетке.

Introduction

Theory of polynomials, orthogonal with respect of Sobolev inner product is intensively developing last three decades, and it has found numerous applications (see [1] – [6] and cited literature). The characteristic feature of Sobolev inner product is that it generally has component that «control» the behavior of corresponding orthogonal polynomials in one or several points of number axis.

For example, it is often considered inner product of type

$$\langle f, g \rangle = \sum_{\nu=0}^{r-1} f^{(\nu)}(a)g^{(\nu)}(a) + \int_a^b f^{(r)}(t)g^{(r)}(t)\rho(t)dt, \quad (0.1)$$

in which f и g – functions, given on $[a, b]$ and continuously differentiable there $r - 1$ times, for which $f^{(r-1)}(x)$ and $g^{(r-1)}(x)$ absolutely continuous and $f^{(r)}(x), g^{(r)}(x) \in L^2_\rho(a, b)$, where $L^2_\rho(a, b)$ – the Lebesgue space with weight $\rho(x)$. It should be noted, that polynomials, orthogonal on Sobolev, by their properties may essentially differ from usual orthogonal on an interval polynomials. For example, in some cases polynomials orthogonal on Sobolev on an interval (a, b) can have zeros, tied with one or both ends of this interval. This circumstance is important for some applications, in which necessary, that values of partial sums of Fourier series of the function $f(x)$ over considered system of orthogonal polynomials would be tied in the ends of the interval (a, b) with values $f(a)$ and $f(b)$. Let us notice, that usual orthogonal polynomials with positive on (a, b) weight don't have this important property. The inner product (0.1) have one special point, name point a , in vicinity of which it is «controlled» the behavior of corresponding polynomials, orthogonal on Sobolev. This is achieved by presence in the inner product (0.1) the summand of appearance $\sum_{\nu=0}^{r-1} f^{(\nu)}(a)g^{(\nu)}(a)$.

On the other hand, in works [7] – [18] independently were considered so-called mixed series on classical orthogonal systems $\{\varphi_k(x)\}$, which represent series on systems of functions $\{\varphi_{r,k}(x)\}$, derived by corresponding system $\{\varphi_k(x)\}$ by application of equalities of the form

$$\varphi_{r,k}(x) = \frac{(x-a)^k}{k!}, \quad k = 0, 1, \dots, r-1 \quad (0.2)$$

and

$$\varphi_{r,k}(x) = \frac{1}{(r-1)!} \int_a^x (x-t)^{r-1} \varphi_{k-r}(t) dt \quad k = r, r+1, \dots \quad (0.3)$$

Let us denote through $W^r_{L^2_\rho(a,b)}$ weighted space of Sobolev, which consists of functions $f(x)$, given on $[a, b]$ and continuously differentiable there $r - 1$ times, for which $f^{(r-1)}(x)$ absolutely continuous and $f^{(r)}(x) \in L^2_\rho(a, b)$. If $f \in W^r_{L^2_\rho(a,b)}$, then mixed series on system $\{\varphi_k(x)\}$ have an appearance

$$f(x) \sim \sum_{k=0}^{r-1} f^{(k)}(a) \frac{(x-a)^k}{k!} + \sum_{k=r}^{\infty} f_{r,k} \varphi_{r,k}(x), \quad (0.4)$$

where

$$f_{r,k} = \int_a^b f^{(r)}(t) \varphi_{r,k}^{(r)}(t) \rho(t) dt = \int_a^b f^{(r)}(t) \varphi_{k-r}(t) \rho(t) dt.$$

Using the definition of functions $\varphi_{r,k}(x)$ (see (0.2) и (0.3)) if is not difficult to see [18], that the system $\{\varphi_{r,k}(x)\}_{k=0}^{\infty}$ is orthogonal in space $W^r_{L^2_\rho(a,b)}$ corresponding to inner product (0.1), values $f^{(k)}(a)$ ($k = 0, 1, \dots, r-1$) and $f_{r,k}$ ($k = r, r+1, \dots$) are Fourier coefficients of the function f , and mixed series (7.3) represent Fourier series of function $f \in W^r_{L^2_\rho(a,b)}$ on this system. We will call this system $\{\varphi_{r,k}(x)\}_{k=0}^{\infty}$ *orthogonal on Sobolev* with respect to inner product (0.1) and derived by orthogonal system $\{\varphi_k(x)\}_{k=0}^{\infty}$. Thus, the notion of mixed series on orthogonal system $\{\varphi_k(x)\}_{k=0}^{\infty}$, introduced in works [7] – [18], coincides with the notion of

Fourier series on system $\{\varphi_{r,k}(x)\}_{k=0}^{\infty}$, orthogonal on Sobolev with respect to inner product (0.1) and derived by system $\{\varphi_k(x)\}_{k=0}^{\infty}$. For the convenience we will (following [7] – [18]) call the Fourier series of appearance (7.3) *mixed series on system* $\{\varphi_k(x)\}_{k=0}^{\infty}$, thinking of this name as of conditional and brief name of the full name: «*Fourier series on the system* $\{\varphi_{r,k}(x)\}_{k=0}^{\infty}$, *orthogonal on Sobolev, derived by orthogonal system* $\{\varphi_k(x)\}_{k=0}^{\infty}$ ».

It should be noted, that mixed series on classical orthogonal polynomials were [7] – [18] as an alternative to Fourier series on the same polynomials instrument of solving tasks, in which you want approximate at the same time differentiable function and several of its derivatives. In particular, this task often occurs when solving differential equations by spectral methods [25], [26]. Mixed series of orthogonal systems of functions are found to be rather effective method of solving boundary value problems for differential equations. On this occasion we can send the reader for example to work [30].

Discrete analogue of inner product (0.1) is of the form

$$\langle f, g \rangle = \sum_{k=0}^{r-1} \Delta^k f(0) \Delta^k g(0) + \sum_{j=0}^{N-1} \Delta^r f(j) \Delta^r g(j) \mu(j), \quad (0.5)$$

where functions f and g given on the set $\Omega_{N+r} = \{0, 1, \dots, N+r-1\}$, $\mu = \mu(j)$ – discrete weighted function, given on the set $\Omega_N = \{0, 1, \dots, N-1\}$. In case, when $r = 0$ we will consider $\sum_{k=0}^{r-1} \Delta^k f(0) \Delta^k g(0) = 0$. When $r \geq 1$, the point $x = 0$ is a singular point in the inner product (0.5), in which the behavior of corresponding orthogonal on Sobolev polynomials of discrete variable is controlled, due to the presence in (0.5) the expression $\sum_{k=0}^{r-1} \Delta^k f(0) \Delta^k g(0)$. In the present article we consider systems of discrete functions, orthogonal on Sobolev with respect to inner product (0.5), derived by the given orthonormal system of functions of discrete variable. The main attention will be given to studying of properties of polynomials, orthogonal on Sobolev, derived by the classical Chebyshev orthogonal polynomials of discrete variable.

1. Systems of discrete functions, orthonormal on Sobolev, derived by orthonormal functions

Let us turn to the constructing of discrete functions, orthogonal on Sobolev with respect to inner product (0.5), derived by the given system $\{\psi_k(x)\}_{k=0}^{N-1}$, orthogonal on the discrete set $\Omega_N = \{0, 1, \dots, N-1\}$ with weight $\mu(x)$. For this we need some notations and concepts. If integer $k \geq 0$, then set $a^{[k]} = a(a-1) \cdots (a-k+1)$, $a^{[0]} = 1$ and consider the next functions

$$\psi_{r,k}(x) = \frac{x^{[k]}}{k!}, \quad k = 0, 1, \dots, r-1, \quad (1.1)$$

$$\psi_{r,k}(x) = \begin{cases} \frac{1}{(r-1)!} \sum_{t=0}^{x-r} (x-1-t)^{[r-1]} \psi_{k-r}(t), & r \leq x \leq N+r-1, \\ 0, & x = 0, 1, \dots, r-1, \end{cases} \quad (1.2)$$

which are defined on the net $\Omega_N = \{0, 1, \dots, N-1+r\}$. Let us consider some important difference properties of systems of functions $\psi_{r,k}(x)$, defined by the

equations (1.1) and (1.2). We introduce an operator of finite difference Δf : $\Delta f(x) = f(x+1) - f(x)$ and set $\Delta^{\nu+1}f(x) = \Delta\Delta^\nu f(x)$. We have the following

LEMMA 1. *We have the equalities*

$$\Delta^\nu \psi_{r,k}(x) = \begin{cases} \psi_{r-\nu,k-\nu}(x), & \text{if } 0 \leq \nu \leq r-1, r \leq k, \\ \psi_{k-r}(x), & \text{if } \nu = r \leq k, \\ \psi_{r-\nu,k-\nu}(x), & \text{if } \nu \leq k < r, \\ 0, & \text{if } k < \nu \leq r. \end{cases} \quad (1.3)$$

PROOF 1. The truth of the Lemma 2.1 when $r = 1$ almost obvious, that is why we consider $r \geq 2$. First of all we notice, that if $f(x) = (x-1-t)^{[r-1]}$, then $\Delta f(x) = (r-1)(x-1-t)^{[r-2]}$, then for $r \leq x, k \leq N+r-1$ due to (1.2) we have

$$\begin{aligned} \Delta \psi_{r,k}(x) &= \frac{1}{(r-1)!} \left(\sum_{t=0}^{x-r+1} (x-t)^{[r-1]} \psi_{k-r}(t) - \sum_{t=0}^{x-r} (x-1-t)^{[r-1]} \psi_{k-r}(t) \right) = \\ &= \frac{1}{(r-1)!} \sum_{t=0}^{x-r+1} \left((x-t)^{[r-1]} - (x-1-t)^{[r-1]} \right) \psi_{k-r}(t) = \\ &= \frac{1}{(r-2)!} \sum_{t=0}^{x-r+1} (x-1-t)^{[r-2]} \psi_{k-1-(r-1)}(t) = \psi_{r-1,k-1}(x). \end{aligned}$$

Hence we verify the validity of the first of equalities (1.3) for $r \leq x \leq N+r-1$. The truth of equality $\Delta \psi_{r,k}(x) = \psi_{r-1,k-1}(x)$ for $x = 0, 1, \dots, r-2$ is obvious. It remains to verify the first of the equalities (1.3) for $x = r-1$. But in this case we have

$$\begin{aligned} \Delta \psi_{r,k}(x) &= \psi_{r,k}(x+1) = \frac{1}{(r-1)!} \sum_{t=0}^{x-r+1} (x-t)^{[r-1]} \psi_{k-r}(t) = \\ &= \frac{(r-1)^{[r-1]}}{(r-1)!} \psi_{k-r}(0) = \psi_{k-r}(0), \\ \psi_{r-1,k-1}(x) &= \frac{1}{(r-2)!} \sum_{t=0}^{x-r+1} (x-1-t)^{[r-2]} \psi_{(k-1)-(r-1)}(t) = \\ &= \frac{(r-2)^{[r-2]}}{(r-2)!} \psi_{k-r}(0) = \psi_{k-r}(0), \end{aligned}$$

Therefore again we find $\Delta \psi_{r,k}(x) = \psi_{r-1,k-1}(x)$. Thus, we fully proved the first of the equations (1.3) for $0 \leq \nu \leq 1$. It's truth for $2 \leq \nu \leq r-1$ we derive by induction. Let us consider the second of the equalities (1.3). Due to the proved first equality (1.3) and the fact, that the second one for $r = 1$ can easily be verified, we have

$$\Delta^r \psi_{r,k}(x) = \Delta \Delta^{r-1} \psi_{r,k}(x) = \Delta \psi_{1,k-r+1}(x) = \psi_{k-r}(x).$$

Thus we proved the second of the equalities (1.3). The third and the fourth of the equalities (1.3) are directly derived from the (1.1). Lemma 1 is fully proved.

Set $0 \leq r$ - integer. Let us denote l_μ^{N+r} the space of discrete function $f = f(x)$, given on the net $\Omega_{N+r} = \{0, 1, \dots, N+r-1\}$, in which inner product $\langle f, g \rangle$ defined using the equation (0.5).

Let us consider the problem of orthogonality, normality and completeness in l_μ^{N+r} of system $\{\psi_{r,k}(x)\}_{k=0}^{N+r-1}$, consisting of functions, defined by the equalities (1.1) and (1.2).

THEOREM 1. *Let us suppose, that functions $\psi_k(x)$ ($k = 0, 1, \dots, N-1$) form a complete in l_μ^N orthonormal system with weight $\mu(x)$. Then system $\{\psi_{r,k}(x)\}_{k=0}^{N+r-1}$, derived by the $\{\psi_k(x)\}_{k=0}^{N-1}$ by equalities (1.1) and (1.2), is complete in l_μ^{N+r} and orthonormal with respect to inner product (0.5).*

PROOF 2. From equality (1.2) it follows, that if $r \leq k$ и $0 \leq \nu \leq r-1$, то $\Delta^\nu \psi_{r,k}(x)_{x=0} = 0$, then due to (0.5) и (1.3) we have

$$\langle \psi_{r,k}, \psi_{r,l} \rangle = \sum_{x=0}^{N-1} \Delta^r \psi_{r,k}(x) \Delta^r \psi_{r,l}(x) \mu(x) =$$

$$\sum_{x=0}^{N-1} \psi_{k-r}(x) \psi_{l-r}(x) \mu(x) = \delta_{kl}, \quad k, l \geq r,$$

$$\langle \psi_{r,k}, \psi_{r,l} \rangle = \sum_{\nu=0}^{r-1} \Delta^\nu \psi_{r,k}(0) \Delta^\nu \psi_{r,l}(0) = \delta_{kl}, \quad k, l < r.$$

Also obvious, that

$$\langle \psi_{r,k}, \psi_{r,l} \rangle = 0, \quad \text{если } k < r \leq l \quad \text{или} \quad l < r \leq k.$$

It means, that functions $\psi_{r,k}(t)$ ($k = 0, 1, \dots, N+r-1$) form in l_μ^{N+r} an orthonormal system with respect to inner product (0.5).

System of functions $\psi_{r,k}(t)$ ($k = 0, 1, \dots, N+r-1$) we will call system, orthonormal on Sobolev with respect to inner product (0.5).

From Theorem 1 follows that system of discrete functions $\psi_{r,k}(t)$ ($k = 0, 1, \dots, N+r-1$) is orthonormal basis in space l_μ^{N+r} , that is why for the arbitrary function $f(x) \in l_\mu^{N+r}$ we can write the equality

$$f(x) = \sum_{k=0}^{N+r-1} \langle f, \psi_{r,k} \rangle \psi_{r,k}(x), \quad (1.4)$$

which if the finite Fourier series of the function $f(x) \in l_\mu^{N+r}$ on the system $\{\psi_{r,k}(t)\}_{k=0}^{N+r-1}$, orthonormal on Sobolev. Since the Fourier coefficients $\langle f, \psi_{r,k} \rangle$ are of the form

$$f_{r,k} = \langle f, \psi_{r,k} \rangle = \sum_{\nu=0}^{r-1} \Delta^\nu f(0) \Delta^\nu \psi_{r,k}(0) = \Delta^k f(0), \quad k = 0, \dots, r-1,$$

$$f_{r,k} = \langle f, \psi_{r,k} \rangle = \sum_{j=0}^{N-1} \Delta^r f(j) \Delta^r \psi_{r,k}(j) \mu(j) = \sum_{j=0}^{N-1} \Delta^r f(j) \psi_{k-r}(j) \mu(j), \quad k = r, \dots, N+r-1, \quad (1.5)$$

we can write the equality (1.4) in mixed form:

$$f(x) = \sum_{k=0}^{r-1} \Delta^k f(0) \frac{x^{[k]}}{k!} + \sum_{k=r}^{N+r-1} f_{r,k} \psi_{r,k}(x), \quad x \in \Omega_{N+r}. \quad (1.6)$$

So Fourier series on the system $\{\psi_{r,k}(t)\}_{k=0}^{N+r-1}$ following [7] – [18] we will call mixed series on $\{\psi_k(t)\}_{k=0}^{N-1}$. We note some important properties of mixed series (1.6) and their partial sums of the form

$$\mathcal{Y}_{r,n}(f, x) = \sum_{k=0}^{r-1} \Delta^k f(0) \frac{x^{[k]}}{k!} + \sum_{k=r}^n f_{r,k} \psi_{r,k}(x). \quad (1.7)$$

From (1.6) and (1.7) taking into account equations (1.3) we may write ($0 \leq \nu \leq r-1$, $x \in \Omega_{N+r-\nu}$)

$$\Delta^\nu f(x) = \sum_{k=0}^{r-\nu-1} \Delta^{k+\nu} f(0) \frac{x^{[k]}}{k!} + \sum_{k=r-\nu}^{N+r-\nu-1} f_{r,k+\nu} \psi_{r-\nu,k}(x), \quad (1.8)$$

$$\Delta^\nu \mathcal{Y}_{r,n}(f, x) = \sum_{k=0}^{r-\nu-1} \Delta^{k+\nu} f(0) \frac{x^{[k]}}{k!} + \sum_{k=r-\nu}^{n-\nu} f_{r,k+\nu} \psi_{r-\nu,k}(x), \quad (1.9)$$

$$\Delta^\nu \mathcal{Y}_{r,n}(f, x) = \mathcal{Y}_{r-\nu, n-\nu}(\Delta^\nu f, x). \quad (1.10)$$

$$\Delta^\nu \mathcal{Y}_{r,n}(f, 0) = \Delta^\nu f(0), \quad 0 \leq \nu \leq r-1. \quad (1.11)$$

In addition, from (1.3) and (1.6) we have

$$\Delta^r f(x) = \sum_{k=0}^{N-1} f_{r,r+k} \psi_k(x), \quad x \in \Omega_N. \quad (1.12)$$

2. Some information about Chebyshev polynomials orthogonal on the uniform net

When we construct polynomials, orthogonal on Sobolev and derived by classical Chebyshev polynomials, orthogonal on a uniform net, we need some properties of these polynomials, which we give in this paragraph. Let N – natural number, α, β – arbitrary numbers. Let

$$\rho(x) = \rho(x; \alpha, \beta, N) = \frac{\Gamma(x + \beta + 1) \Gamma(N - x + \alpha)}{\Gamma(x + 1) \Gamma(N - x)}, \quad (2.1)$$

$$T_n^{\alpha, \beta}(x, N) = \frac{(-1)^n}{n! (N-1)^{[n]} \rho(x)} \Delta^n \left\{ \rho(x) (x - N - \alpha)^{[n]} x^{[n]} \right\}, \quad (2.2)$$

where $\Delta^n f(x)$ – finite difference of the order n of the function $f(x)$ in the point x , i.e. $\Delta^0 f(x) = f(x)$, $\Delta^1 f(x) = \Delta f(x) = f(x+1) - f(x)$, $\Delta^n f(x) = \Delta \Delta^{n-1} f(x)$ ($n \geq 1$), $a^{[0]} = 1$, $a^{[k]} = a(a-1) \cdots (a-k+1)$ for $k \geq 1$. For each $0 \leq n \leq N-1$ equality (2.2) defines [23] algebraic polynomial of degree n , for which

$$T_n^{\alpha, \beta}(N-1, N) = \binom{n+\alpha}{n}, \quad T_n^{\alpha, \beta}(0, N) = (-1)^n \binom{n+\beta}{n}.$$

Full proofs of given below properties of the Chebyshev polynomials $T_n^{\alpha, \beta}(x, N)$ can be found, for example in [29]. First of all, let us note, that polynomials $T_n^{\alpha, \beta}(x, N)$ can be presented in the following form

$$T_n^{\alpha, \beta}(x, N) = (-1)^n \frac{\Gamma(n+\beta+1)}{n!} \sum_{k=0}^n (-1)^k \frac{n^{[k]}(n+\alpha+\beta+1)_k x^{[k]}}{\Gamma(k+\beta+1)k!(N-1)^{[k]}}. \quad (2.3)$$

If $\alpha, \beta > -1$, then polynomials $T_n^{\alpha, \beta}(x, N)$ ($0 \leq n \leq N-1$) form orthogonal with weight $\rho(x)$ (see (2.1)) system on the set $\Omega_N = \{0, 1, \dots, N-1\}$, more precisely

$$\sum_{x \in \Omega_N} \mu(x) T_n^{\alpha, \beta}(x, N) T_m^{\alpha, \beta}(x, N) = h_{n, N}^{\alpha, \beta} \delta_{nm}, \quad (2.4)$$

where δ_{nm} – Kronecker delta,

$$\begin{aligned} \mu(x) &= \mu(x; \alpha, \beta, N) = \frac{\Gamma(N)2^{\alpha+\beta+1}}{\Gamma(N+\alpha+\beta+1)} \rho(x) \\ &= \frac{\Gamma(N)2^{\alpha+\beta+1}}{\Gamma(N+\alpha+\beta+1)} \frac{\Gamma(x+\beta+1)\Gamma(N-x+\alpha)}{\Gamma(x+1)\Gamma(N-x)}, \\ h_{n, N}^{\alpha, \beta} &= \frac{(N+n+\alpha+\beta)^{[n]}}{(N-1)^{[n]}} \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)2^{\alpha+\beta+1}}{n!\Gamma(n+\alpha+\beta+1)(2n+\alpha+\beta+1)}. \end{aligned} \quad (2.5)$$

When $n=0$ the product $(\alpha+\beta+1)\Gamma(\alpha+\beta+1)$ can be replaced with $\Gamma(\alpha+\beta+2)$. For $0 \leq n \leq N-1$ we note

$$\tau_n^{\alpha, \beta}(x) = \tau_n^{\alpha, \beta}(x, N) = \left\{ h_{n, N}^{\alpha, \beta} \right\}^{-1/2} T_n^{\alpha, \beta}(x, N). \quad (2.6)$$

Obviously, if $0 \leq n, m \leq N-1$, then

$$\sum_{x=0}^{N-1} \mu(x) \tau_n^{\alpha, \beta}(x, N) \tau_m^{\alpha, \beta}(x, N) = \delta_{nm}. \quad (2.7)$$

In other words, polynomials $\tau_n^{\alpha, \beta}(x, N)$ ($0 \leq n \leq N-1$) form orthogonal with weight $\mu(x)$ system on Ω_N .

The Cristoffel-Darboux formula for $\tau_n^{\alpha, \beta}(x, N)$ has the next appearance

$$\begin{aligned} \mathcal{K}_{n, N}^{\alpha, \beta}(x, y) &= \sum_{k=0}^n \tau_k^{\alpha, \beta}(x) \tau_k^{\alpha, \beta}(y) = \sum_{k=0}^n \frac{T_k^{\alpha, \beta}(x) T_k^{\alpha, \beta}(y)}{h_{k, N}^{\alpha, \beta}} = \\ &= \frac{(N-1)^{[n+1]}}{(N+n+\alpha+\beta)^{[n]}} \frac{2^{-\alpha-\beta-1}}{2n+\alpha+\beta+2} \frac{\Gamma(n+2)\Gamma(n+\alpha+\beta+2)}{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)} \times \end{aligned}$$

$$\frac{T_{n+1}^{\alpha,\beta}(x)T_n^{\alpha,\beta}(y) - T_n^{\alpha,\beta}(x)T_{n+1}^{\alpha,\beta}(y)}{x - y}. \quad (2.8)$$

Since $\Delta a^{[k]} = ka^{[k-1]}$, from (2.3) we found

$$\begin{aligned} & (n+1)T_{n+1}^{\alpha,\beta}(x, N) + (n+\beta+1)T_n^{\alpha,\beta}(x, N) \\ &= \frac{2n+\alpha+\beta+2}{N-1}xT_n^{\alpha,\beta+1}(x-1, N-1). \end{aligned} \quad (2.9)$$

From equality $\mu(N-1-x; \beta, \alpha, N) = \mu(x; \alpha, \beta, N)$, that directly follows from orthogonality (2.4), when $\alpha, \beta > -1$ follows the equality

$$T_n^{\alpha,\beta}(x, N) = (-1)^n T_n^{\beta,\alpha}(N-1-x, N). \quad (2.10)$$

Since both parts of this equality are analytic with respect to α and β , it is valid for arbitrary α and β . From (2.9) and (2.10) we have also the next equality

$$\begin{aligned} & (n+\alpha+1)T_n^{\alpha,\beta}(x, N) - (n+1)T_{n+1}^{\alpha,\beta}(x, N) \\ &= \frac{2n+\alpha+\beta+2}{N-1}(N-1-x)T_n^{\alpha+1,\beta}(x, N-1). \end{aligned}$$

Directly from explicit formula (2.3) we can derive the following useful equation

$$\Delta^m T_n^{\alpha,\beta}(x, N) = \frac{(n+\alpha+\beta+1)_m}{(N-1)^{[m]}} T_{n-m}^{\alpha+m,\beta+m}(x, N-m), \quad (2.11)$$

where $(a)_0 = 1$, $(a)_k = a(a+1)\cdots(a+k-1)$ при $k \geq 1$. If β such integer, that $-n \leq \beta \leq -1$, then from (2.11) we can derive also

$$T_n^{\alpha,\beta}(x, N) = \frac{(n+\beta)!}{n!} \frac{(n+\alpha)^{[-\beta]} x^{[-\beta]}}{(N-1)^{[-\beta]}} T_{n+\beta}^{\alpha,-\beta}(x+\beta, N+\beta),$$

and if α and β – integers, $-n \leq \beta \leq -1$, $-(n+\beta) \leq \alpha \leq -1$, $N \geq 2$, then

$$T_n^{\alpha,\beta}(x, N) = \frac{(-1)^\alpha x^{[-\beta]} (N-x-1)^{[-\alpha]}}{(N-1)^{[-\beta]} (N-1+\beta)^{[-\alpha]}} T_{n+\alpha+\beta}^{-\alpha,-\beta}(x+\beta, N+\alpha+\beta). \quad (2.12)$$

Difference Rodrigues' formula (2.1) allows the following generalization

$$\begin{aligned} & \rho(x+m; \alpha, \beta, N+m) T_n^{\alpha,\beta}(x+m, N+m) = \\ & \frac{(-1)^m}{n^{[m]}(N)_m} \Delta^m \left\{ \rho(x; \alpha+m, \beta+m, N) T_{n-m}^{\alpha+m,\beta+m}(x, N) \right\}, \end{aligned}$$

which directly follows from (2.1). Replacing here m by ν , α and β by $m-\nu$, n by $k+\nu-m$, we can also write

$$\begin{aligned} & \Delta^\nu \{ (x+1)_m (N-x)_m T_{k-m}^{m,m}(x, N) \} = \\ & (-1)^\nu (k+\nu-m)^{[\nu]} (N+\nu-1)^{[\nu]} (x+1+\nu)_{m-\nu} (N-x)_{m-\nu} T_{k+\nu-m}^{m-\nu,m-\nu}(x+\nu, N+\nu). \end{aligned}$$

If in the equality (2.11) we replace α, β and n , correspondingly, by $\alpha - m, \beta - m$ и $k + m$, then we come to formula

$$\Delta^m T_{k+m}^{\alpha-m, \beta-m}(x, N) = \frac{(k + \alpha + \beta)^{[m]}}{(N - 1)^{[m]}} T_k^{\alpha, \beta}(x, N - m). \quad (2.13)$$

When $\alpha, \beta > -1$ polynomials $\tau_n^{\alpha, \beta}(x, N)$, $n = 2, 3, \dots, N - 1$, are connected together with recurrent formula, which in case $\alpha = \beta$ is of the form:

$$\tau_n^{\alpha, \alpha}(x, N) = \hat{\kappa}_n^{\alpha, \alpha}(2x - N + 1) \tau_{n-1}^{\alpha, \alpha}(x, N) - \hat{\mu}_n^{\alpha, \alpha} \tau_{n-2}^{\alpha, \alpha}(x, N). \quad (2.14)$$

3. Polynomials, orthogonal on Sobolev, derived by Chebyshev polynomials, orthogonal on the uniform net

From equality (2.7) follows, that if $\alpha, \beta > -1$, then polynomials $\tau_n^{\alpha, \beta}(x, N)$ ($n = 0, 1, \dots, N - 1$) form orthonormal on $\Omega_N = \{0, 1, \dots, N - 1\}$ with weight $\mu(x)$ (see (2.5)) system. This system derive for $x \in \{r, r + 1, \dots, N - 1 + r\}$ system of polynomials $\tau_{r, k+r}^{\alpha, \beta}(x, N)$ ($k = 0, 1, \dots, N - 1$), defined by the equality

$$\tau_{r, k+r}^{\alpha, \beta}(x, N) = \frac{1}{(r - 1)!} \sum_{t=0}^{x-r} (x - 1 - t)^{[r-1]} \tau_k^{\alpha, \beta}(t, N). \quad (3.1)$$

Besides, we define polynomials

$$\tau_{r, k}^{\alpha, \beta}(x, N) = \frac{x^{[k]}}{k!}, \quad k = 0, 1, \dots, r - 1. \quad (3.2)$$

Let us show, that polynomial $\tau_{r, k+r}^{\alpha, \beta}(x, N)$ is zero in nodes $x \in \{0, 1, \dots, r - 1\}$. For this purpose we consider the next discrete analogue of the Taylor's formula ($x \in \{r, r + 1, \dots, N - 1 + r\}$)

$$F(x) = Q_{r-1}(F, x) + \frac{1}{(r - 1)!} \sum_{t=0}^{x-r} (x - 1 - t)^{[r-1]} \Delta^r F(t), \quad (3.3)$$

where

$$Q_{r-1}(F, x) = F(0) + \frac{\Delta F(0)}{1!} x + \frac{\Delta^2 F(0)}{2!} x^{[2]} + \dots + \frac{\Delta^{r-1} F(0)}{(r - 1)!} x^{[r-1]}. \quad (3.4)$$

Since for the function $F(x) = x^{[l+r]}$, where integer $l \geq 0$, we have $\Delta^r F(x) = (l + r)^{[r]} x^{[l]}$ и $Q_{r-1}(F, x) \equiv 0$, then from (3.3) follows, that

$$\begin{aligned} & \frac{1}{(r - 1)!} \sum_{t=0}^{x-r} (x - 1 - t)^{[r-1]} t^{[l]} = \\ & \frac{1}{(l + r)^{[r]} (r - 1)!} \sum_{t=0}^{x-r} (x - 1 - t)^{[r-1]} \Delta^r F(t) = \frac{x^{[l+r]}}{(l + r)^{[r]}}. \end{aligned} \quad (3.5)$$

On the other hand, the function $x^{[l+r]}$ is zero in nodes $x \in \{0, 1, \dots, r - 1\}$, therefore our statement follows from the fact, that polynomial $\tau_{k+r}^{\alpha, \beta}(x, N)$ by (2.3) and (2.6) can be represented as linear combination of functions of kind x^l . Therefore, from theorem 1 and property (2.7) we directly derive the following result.

TEOPEMA 1. If $\alpha, \beta > -1$, then the system of polynomials $\tau_{r,k}^{\alpha,\beta}(x, N)$ ($k = 0, 1, \dots, N-1+r$), derived by Chebyshev polynomials $\tau_n^{\alpha,\beta}(x, N)$ ($n = 0, 1, \dots, N-1$) by equalities (3.1) and (3.2), is complete in l_μ^{N+r} and orthonormal with respect to inner product (0.5).

4. Further properties of polynomials $\tau_{r,k}^{\alpha,\beta}(x, N)$

Now we investigate further properties of polynomials $\tau_{r,k}^{\alpha,\beta}(x, N)$ ($k = 0, 1, \dots, N-1+r$). First of all, we talk about getting the representation of polynomials $\tau_{r,k}^{\alpha,\beta}(x, N)$, which does not contain the sum signs with variable upper bound of type (3.1). For this purpose we apply formula (3.3) to the polynomial $F(x) = T_{k+r}^{\alpha-r, \beta-r}(x, N+r)$ and write

$$F(x) = Q_{r-1}(F, x) + \frac{1}{(r-1)!} \sum_{t=0}^{x-r} (x-1-t)^{[r-1]} \Delta^r T_{k+r}^{\alpha-r, \beta-r}(t, N+r).$$

Instead of $\Delta^r T_{k+r}^{\alpha-r, \beta-r}(x, N+r)$ put its value, which by formula (2.13) equals $\frac{(k+\alpha+\beta)^{[r]}}{(N-1+r)^{[r]}} T_k^{\alpha,\beta}(x, N)$, then from (3.3) we get

$$F(x) - Q_{r-1}(F, x) = \frac{(k+\alpha+\beta)^{[r]}}{(N-1+r)^{[r]}} \frac{1}{(r-1)!} \sum_{t=0}^{x-r} (x-1-t)^{[r-1]} T_k^{\alpha,\beta}(t, N). \quad (4.1)$$

Comparing (3.1) and (4.1) with (2.6), we find

$$\frac{(k+\alpha+\beta)^{[r]}}{(N-1+r)^{[r]}} \left\{ h_{k,N}^{\alpha,\beta} \right\}^{1/2} \tau_{r,k+r}^{\alpha,\beta}(x, N) = F(x) - Q_{r-1}(F, x). \quad (4.2)$$

If $(k+\alpha+\beta)^{[r]} \neq 0$, then from (4.2) we have $\left(F(x) = T_{k+r}^{\alpha-r, \beta-r}(x, N+r) \right)$

$$\tau_{r,k+r}^{\alpha,\beta}(x, N) = \frac{(N-1+r)^{[r]}}{(k+\alpha+\beta)^{[r]}} \left\{ h_{k,N}^{\alpha,\beta} \right\}^{-1/2} [F(x) - Q_{r-1}(F, x)]. \quad (4.3)$$

Further, by (2.11)

$$\Delta^m T_{k+r}^{\alpha-r, \beta-r}(0, N+r) = \frac{(k+\alpha+\beta-r+1)_m}{(N-1+r)^{[m]}} T_{k+r-m}^{\alpha-r+m, \beta-r+m}(0, N+r-m), \quad (4.4)$$

and from (2.3) we have

$$T_{k+r-m}^{\alpha-r+m, \beta-r+m}(0, N+r-m) = \frac{(-1)^{k+r-m} \Gamma(k+\beta+1)}{\Gamma(m-r+\beta+1)(k+r-m)!}. \quad (4.5)$$

From (4.4) and (4.5) follows, that ($0 \leq m \leq r-1$)

$$\begin{aligned} A_{r,k,m} &= A_{r,k,m}(\alpha, \beta) = \Delta^m T_{k+r}^{\alpha-r, \beta-r}(0, N+r) = \\ &= \frac{(k+\alpha+\beta-r+1)_m}{(N-1+r)^{[m]}} \frac{(-1)^{k+r-m} \Gamma(k+\beta+1)}{\Gamma(m-r+\beta+1)(k+r-m)!}. \end{aligned} \quad (4.6)$$

Equalities (3.4) and (4.6), taken together, give us

$$F(x) - Q_{r-1}(F, x) = T_{k+r}^{\alpha-r, \beta-r}(x, N+r) - \sum_{m=0}^{r-1} \frac{A_{r,k,m} x^{[m]}}{m!}.$$

Lets put this expression to (4.3), then in the case, when $(k + \alpha + \beta)^{[r]} \neq 0$ we find

$$\tau_{r,k+r}^{\alpha, \beta}(x, N) = \frac{(N-1+r)^{[r]}}{(k + \alpha + \beta)^{[r]} \left\{ h_{k,N}^{\alpha, \beta} \right\}^{\frac{1}{2}}} \left[T_{k+r}^{\alpha-r, \beta-r}(x, N+r) - \sum_{m=0}^{r-1} \frac{A_{r,k,m} x^{[m]}}{m!} \right]. \quad (4.7)$$

One more important representation for the polynomials $\tau_{r,k+r}^{\alpha, \beta}(x, N)$ can be obtained, if we if we turn to equalities (2.3) and (2.6) and write

$$\tau_k^{\alpha, \beta}(x, N) = (-1)^k \frac{\Gamma(k + \beta + 1)}{k! \left\{ h_{k,N}^{\alpha, \beta} \right\}^{1/2}} \sum_{l=0}^k (-1)^l \frac{k^{[l]} (k + \alpha + \beta + 1)_l x^{[l]}}{\Gamma(l + \beta + 1) l! (N-1)^{[l]}}. \quad (4.8)$$

Lets put this expression to (3.1) and use the equality (3.5). This leads us to the following explicit form for polynomials $\tau_{r,k+r}^{\alpha, \beta}(x, N)$:

$$\tau_{r,k+r}^{\alpha, \beta}(x, N) = (-1)^k \frac{\Gamma(k + \beta + 1)}{k! \left\{ h_{k,N}^{\alpha, \beta} \right\}^{1/2}} \sum_{l=0}^k (-1)^l \frac{k^{[l]} (k + \alpha + \beta + 1)_l x^{[l+r]}}{\Gamma(l + \beta + 1) (N-1)^{[l]} (l+r)!}. \quad (4.9)$$

5. Some important special cases of system $\{\tau_{r,k}^{\alpha, \beta}(x, N)\}_{k=0}^{N-1+r}$

Lets consider some special cases of systems $\{\tau_{r,k}^{\alpha, \beta}(x, N)\}_{k=0}^{N-1+r}$, corresponding to values of parameters α и β , chosen with the next two ways: 1) $\alpha = \beta = 0$; 2) α – fractional, $\beta = 0$.

5.1. System $\{\tau_{r,k}^{0,0}(x, N)\}_{k=0}^{N-1+r}$. Let us note, that if $\beta = \alpha = 0$, to $(k + \alpha + \beta)^{[r]} = k^{[r]} \neq 0$ for each $k \geq r$, and from (4.6) follows that $A_{r,k,m}(0, 0) = 0$ for all $m = 0, 1, \dots, r-1$. Therefore for $r \leq k \leq N-1$ from (4.7) we have

$$\tau_{r,k+r}^{0,0}(x, N) = \frac{(N-1+r)^{[r]}}{k^{[r]} \left\{ h_{k,N}^{0,0} \right\}^{\frac{1}{2}}} T_{k+r}^{-r, -r}(x, N+r). \quad (5.1)$$

Further, if we look at inequality (2.12), then we can write

$$T_{k+r}^{-r, -r}(x, N+r) = \frac{(-1)^r x^{[r]} (N-1+r-x)^{[r]}}{(N-1+r)^{[r]} (N-1)^{[r]}} T_{k-r}^{r, r}(x-r, N-r). \quad (5.2)$$

From (5.1), (5.2) and (2.6) we find for $r \leq k \leq N-1$

$$\tau_{r,r+k}^{0,0}(x) = \frac{(-1)^r x^{[r]} (N-1+r-x)^{[r]}}{(N-1)^{[r]} k^{[r]} \left\{ h_{k,N}^{0,0} \right\}^{\frac{1}{2}}} T_{k-r}^{r, r}(x-r, N-r) =$$

$$\frac{(-1)^r x^{[r]} (N-1+r-x)^{[r]}}{(N-1)^{[r]} k^{[r]}} \left\{ \frac{h_{k-r, N-r}^{r,r}}{h_{k, N}^{0,0}} \right\}^{\frac{1}{2}} \tau_{k-r}^{r,r}(x-r, N-r). \quad (5.3)$$

Regarding polynomials $\tau_{r, k+r}^{0,0}(x, N)$ $0 \leq k \leq r-1$, by (4.9)

$$\tau_{r, k+r}^{0,0}(x, N) = \frac{(-1)^k}{\{h_{k, N}^{0,0}\}^{1/2}} \sum_{l=0}^k \frac{(-1)^l k^{[l]} (k+1)_l x^{[l+r]}}{l! (N-1)^{[l]} (l+r)!}.$$

Finally, if $0 \leq k \leq r-1$, then by definition (3.2)

$$\tau_{r, k}^{0,0}(x, N) = \frac{x^{[k]}}{k!}. \quad (5.4)$$

From theorem 1 follows, that system of polynomials $\tau_{r, k}^{0,0}(x, N)$ ($k = 0, 1, \dots, N+r-1$) is orthonormal basis in the space l_μ^{N+r} , therefore for the arbitrary function $f(x) \in l_\mu^{N+r}$ we can write the equality

$$f(x) = \sum_{k=0}^{N+r-1} \langle f, \tau_{r, k}^{0,0} \rangle \tau_{r, k}^{0,0}(x, N), \quad (5.5)$$

which represent finite Fourier series of function $f(x) \in l_\mu^{N+r}$ on the system $\{\tau_{r, k}^{0,0}(x, N)\}_{k=0}^{N+r-1}$, orthonormal on Sobolev with respect to the inner product (0.5). Since Fourier coefficients $\langle f, \tau_{r, k}^{0,0} \rangle$ are of the form

$$\begin{aligned} f_{r, k} &= \langle f, \tau_{r, k}^{0,0} \rangle = \sum_{\nu=0}^{r-1} \Delta^\nu f(0) \Delta^\nu \tau_{r, k}^{0,0}(0, N) = \Delta^k f(0), \quad k = 0, \dots, r-1, \\ f_{r, k} &= \langle f, \tau_{r, k}^{0,0} \rangle = \sum_{j=0}^{N-1} \Delta^r f(j) \tau_{k-r}^{0,0}(j, N) \mu(j), \quad k = r, \dots, N+r-1, \end{aligned}$$

the equality (5.5) may be written in the next mixed form

$$f(x) = \sum_{k=0}^{r-1} \Delta^k f(0) \frac{x^{[k]}}{k!} + \sum_{k=r}^{N+r-1} f_{r, k} \tau_{r, k}^{0,0}(x, N), \quad x \in \Omega_{N+r}. \quad (5.6)$$

5.2. System $\{\tau_{r, k}^{\alpha,0}(x, N)\}_{k=0}^{N-1+r}$. If α - fractional, and $\beta = 0$, then $(k + \alpha + \beta)^{[r]} = (k + \alpha)^{[r]} \neq 0$ for all $k \geq 0$, and from (4.6) follows that $A_{r, k, m}(\alpha, 0) = 0$ for all $m = 0, 1, \dots, r-1$. Therefore for $0 \leq k \leq N-1$ from (4.7) we have

$$\tau_{r, k+r}^{\alpha,0}(x, N) = \frac{(N-1+r)^{[r]}}{(k+\alpha)^{[r]} \{h_{k, N}^{\alpha,0}\}^{\frac{1}{2}}} T_{k+r}^{\alpha-r, -r}(x, N+r).$$

In addition, if $0 \leq k \leq r-1$, then, by definition (3.2)

$$\tau_{r, k}^{\alpha,0}(x, N) = \frac{x^{[k]}}{k!}.$$

Finite Fourier series on the system $\{\tau_{r,k}^{\alpha,0}(x, N)\}_{k=0}^{N-1+r}$ are of the form

$$f(x) = \sum_{k=0}^{r-1} \Delta^k f(0) \frac{x^{[k]}}{k!} + \sum_{k=r}^{N+r-1} f_{r,k} \tau_{r,k}^{\alpha,0}(x, N), \quad x \in \Omega_{N+r}.$$

where

$$f_{r,k} = \langle f, \tau_{r,k}^{\alpha,0} \rangle = \sum_{j=0}^{N-1} \Delta^r f(j) \tau_{k-r}^{\alpha,0}(j, N) \mu(j), \quad k = r, \dots, N+r-1.$$

6. Difference properties of partial Fourier sums on the system $\{\tau_{r,k}^{\alpha,\beta}(x, N)\}_{k=0}^{N-1+r}$

Main difference properties of Fourier sums on polynomials $\tau_{r,k}^{\alpha,\beta}(x, N)$, which by (1.7) are of the form

$$\mathcal{Y}_{r,n}^{\alpha,\beta}(f, x) = \sum_{k=0}^{r-1} \Delta^k f(0) \frac{x^{[k]}}{k!} + \sum_{k=r}^n f_{r,k} \tau_{r,k}^{\alpha,\beta}(x, N),$$

where

$$f_{r,k} = \langle f, \tau_{r,k}^{\alpha,\beta} \rangle = \sum_{j=0}^{N-1} \Delta^r f(j) \tau_{k-r}^{\alpha,\beta}(j, N) \mu(j), \quad k = r, \dots, N+r-1.$$

are expressed by equalities (1.8) – (1.10). For the system $\{\tau_{r,k}^{\alpha,\beta}(x, N)\}_{k=0}^{N-1+r}$ they take the form ($0 \leq \nu \leq r-1$)

$$\begin{aligned} \Delta^\nu f(x) &= \sum_{k=0}^{r-\nu-1} \Delta^{k+\nu} f(0) \frac{x^{[k]}}{k!} + \sum_{k=r-\nu}^{N+r-\nu-1} f_{r,k+\nu} \tau_{r-\nu,k}^{\alpha,\beta}(x, N), \\ \Delta^\nu \mathcal{Y}_{r,n}^{\alpha,\beta}(f, x) &= \sum_{k=0}^{r-\nu-1} \Delta^{k+\nu} f(0) \frac{x^{[k]}}{k!} + \sum_{k=r-\nu}^{n-\nu} f_{r,k+\nu} \tau_{r-\nu,k}^{\alpha,\beta}(x, N), \\ \Delta^\nu \mathcal{Y}_{r,n}^{\alpha,\beta}(f, x) &= \mathcal{Y}_{r-\nu,n-\nu}^{\alpha,\beta}(\Delta^\nu f, x). \end{aligned}$$

From (1.8) and (1.9) we also can write for $n \geq r > \nu \geq 0$

$$\Delta^\nu f(x) - \Delta^\nu \mathcal{Y}_{r,n}^{\alpha,\beta}(f, x) = \sum_{k=n-\nu+1}^{N+r-\nu-1} f_{r,k+\nu} \tau_{r-\nu,k}^{\alpha,\beta}(x, N). \quad (6.1)$$

Equality (6.1) gives us an expression for an error, resultant by replacement of finite difference $\Delta^\nu f(x)$ with its approximate value $\Delta^\nu \mathcal{Y}_{r,n}^{\alpha,\beta}(f, x)$. When we solve a task of evaluation of the error, the question of asymptotic properties of polynomials $\tau_{r-\nu,k}^{\alpha,\beta}(x, N)$ occurs. This question, in turn, reduces, as shown in paragraphs 5 and 6 of present article, to the task of asymptotic properties of Chebyshev polynomials $\tau_k^{\alpha,\beta}(x, N)$, which were investigated in works [28], [29]. Using results, achieved in works [28], [29] and Christoffel-Darboux formula (2.8), in work [14] the task of

estimating the deviation $|\Delta^\nu f(x) - \Delta^\nu \mathcal{Y}_{r,n}^{\alpha,\beta}(f, x)|$ in case, when $\alpha = \beta = \nu = 0$. Also we note, that mixed series on Chebyshev polynomials $\tau_k^{\alpha,\beta}(x, N)$ are investigated in works [7], [9] и [12], in case $\alpha = \beta = 0$ in essence coincide with Fourier series (5.6) on the system $\{\tau_{r,k}^{0,0}(x, N)\}_{k=0}^{N-1+r}$ (up to a certain designations). Therefore, results, obtained in cited works [7], [9] и [12], concerning approximation of discrete function with partial sums of mixed series on the Chebyshev polynomials $\tau_k^{\alpha,\beta}(x, N)$, directly can be used for estimating of the deviation $|\Delta^\nu f(x) - \Delta^\nu \mathcal{Y}_{r,n}^{0,0}(f, x)|$. The same remark is valid in the case, when α – fractional, and $\beta = 0$.

7. The presentation of solution of the Cauchy problem for difference equation as Fourier series on functions orthogonal on Sobolev

In the present section we consider the question of presentation of solution of the Cauchy problem of difference equation

$$\sum_{l=0}^r a_l(j) \Delta^l y(j) = f(j), \quad j \in \Omega_N = \{0, 1, \dots, N-1\}, \quad (7.1)$$

with initial conditions $\Delta^l y(0) = y_l$, $l = 0, 1, \dots, r-1$, by expanding $y(j)$ on the net $\Omega_{N+r} = \{0, 1, \dots, N-1+r\}$ in a finite Fourier series on polynomials, orthogonal in Sobolev sense the net Ω_{N+r} , where functions a_l , $l = 0, 1, \dots, r-1$, are given on the set Ω_N , and $\Delta^l y$ – operator of the finite difference of the order l . This task is interesting itself, and also because to this task can be reduced the problem of approximated solution of the cauchy problem for the ordinary differential equation of the appearance

$$\sum_{l=0}^r a_l(t) y^{(l)}(t) = f(t)$$

with initial conditions $y^{(l)}(0) = y_l$, $l = 0, 1, \dots, r-1$. Let us notice, that equation (7.1) can be rewritten in the next recurrent form

$$a_r(j) y(j+r) = \sum_{l=0}^{r-1} b_l(j) y(j+l) + f(j), \quad j \in \Omega_N, \quad (7.2)$$

in which $b_l(j)$, $l = 0, 1, \dots, r-1$, – given functions, defined on the net Ω_N . If function $|a_r(j)| \geq c > 0$, $j \in \Omega_N$, then precise solution of the equation (7.1) can be found by using recurrent formula (7.2). If for some $j \in \Omega_N$ we have $a_r(j) = 0$, then it is impossible to find unequivocally values corresponding values $y(j+r)$ by equality (7.2). Besides, let us notice, that if values $f(j)$ of the function f , which appears on the right hand of the equality (7.1) contain errors, and amount of them is big, then using of the recurrent formula (7.2) for finding of solution of the problem $y = y(j)$ can give unsatisfying results even if $|a_r(j)| \geq c > 0$, $j \in \Omega_N$. Thus, we have got the task of searching alternative methods of solving the equation (7.1).

We propose a new method of approximate solving of the Cauchy problem (7.1), based on an application of polynomials $\tau_{r,n}^{\alpha,\beta}(x, N)$, $\alpha, \beta > -1$, $n \in \Omega_{N+r}$, orthogonal

on Sobolev with respect to an inner product of the form

$$\langle f, g \rangle = \sum_{\nu=0}^{r-1} \Delta^\nu f(0) \Delta^\nu g(0) + \sum_{j=0}^{N-1} \Delta^r f(j) \Delta^r g(j) \mu(j), \quad (7.3)$$

where $\mu(x)$ – weight function, defined by the equality (2.5). Polynomials $\tau_{r,k+r}^{\alpha,\beta}(x, N)$ are derived by classical Chebyshev polynomials $\tau_n^{\alpha,\beta}(x, N)$, $n \in \Omega_N$, orthonormal on the net Ω_N with weight $\mu(x)$, and are defined by the equalities (3.1) for $r < k \leq N + r - 1$ and (3.2) for $0 \leq k \leq r - 1$.

One of effective approaches of solving different equations of various types (differential, integral, difference) is [24, 25, 26] so called spectral method, based on a presentation of desired solution as series on a suitable orthonormal system of functions and later conversion of this series to the dual appearance, in which instead of desired solution appear unknown coefficients of its expansion in series on chosen orthonormal system. We will return to the Cauchy problem (7.1) and, using expansion (1.6), represent its desired solution y as a finite Fourier series

$$y(x) = \sum_{k=0}^{r-1} \Delta^k y(0) \frac{x^{[k]}}{k!} + \sum_{k=0}^{N-1} y_{r,k+r} \psi_{r,k+r}(x), \quad x \in \Omega_{N+r}, \quad (7.4)$$

on the system $\{\psi_{r,k}(x)\}_{k=0}^{N+r-1}$, consisting of functions, defined by the equalities (1.1) and (1.2), orthonormal on the net Ω_{N+r} on Sobolev with respect to the inner product (7.3). In addition, by virtue of (1.8) for $x \in \Omega_{N+r-l}$ and $0 \leq l \leq r - 1$ we have

$$\Delta^l y(x) = \sum_{k=0}^{r-l-1} \Delta^{k+l} y(0) \frac{x^{[k]}}{k!} + \sum_{k=0}^{N-1} y_{r,r+k} \psi_{r-l,r-l+k}(x),$$

and from (1.12) follows, that

$$\Delta^r y(x) = \sum_{k=0}^{N-1} y_{r,r+k} \psi_k(x), \quad x \in \Omega_N, \quad (7.5)$$

where, for the coefficients $y_{r,k+r}$ according to (1.5) the following equality holds:

$$y_{r,k+r} = \langle y, \psi_{r,k+r} \rangle = \sum_{j=0}^{N-1} \Delta^r y(j) \psi_k(j) \mu(j), \quad k = 0, \dots, N - 1.$$

Let us notice, that if we find the coefficients $y_{r,k}$ from (7.4), and function $y(x)$ occurs solution of the difference equation (7.1), then, obviously, we obtain exactly the solution, that satisfy our initial conditions $\Delta^l y(0) = y_l$, $l = 0, 1, \dots, r - 1$. In other words, we obtain the decision of the Cauchy problem. It is important to note that the partial sum of the finite Fourier series (7.4) of a form (see (1.7))

$$\mathcal{Y}_{r,n}(y, x) = \sum_{k=0}^{r-1} \Delta^k y(0) \frac{x^{[k]}}{k!} + \sum_{k=0}^n y_{r,k+r} \psi_{r,k+r}(x)$$

with $n \geq r$ also satisfy (see (1.11)) the initial conditions of the Cauchy problem (7.1) and therefore can be considered as approximate solution of this problem.

Concerning this, there is a question about estimating the deviation of approximate solution $\mathcal{Y}_{r,n}(y, x)$ of the Cauchy problem (7.1) from its exact solution $y(x)$, representable in form (7.4), in other words, the task of estimating the remainder $|y(x) - \mathcal{Y}_{r,n}(y, x)|$, $x \in \Omega_{N+r}$ occurs. During solving this task, in particular the equality (6.1) can be used. We will not dwell here on detailed analysis of this problem.

To complete the transition from the equation (7.1) to its dual (spectral) form, We substitute in the left-hand side of the equality instead of the finite differences $\Delta^l y(j)$ their representations from (7.4)–(7.5) and, as a result, we have got an equality

$$a_r(x) \sum_{i=0}^{N-1} y_{r,r+i} \psi_i(x) + \sum_{l=0}^{r-1} a_l(x) \sum_{i=0}^{N-1} y_{r,r+i} \psi_{r-l,r-l+i}(x) =$$

$$f(x) - \sum_{l=0}^{r-1} a_l(x) \sum_{k=0}^{r-l-1} \Delta^{k+l} y(0) \frac{x^{[k]}}{k!},$$

which can be rewritten as

$$\sum_{i=0}^{N-1} y_{r,r+i} \left[a_r(x) \psi_i(x) + \sum_{l=0}^{r-1} a_l(x) \psi_{r-l,r-l+i}(x) \right] =$$

$$f(x) - \sum_{l=0}^{r-1} a_l(x) \sum_{k=0}^{r-l-1} \Delta^{k+l} y(0) \frac{x^{[k]}}{k!}. \quad (7.6)$$

Let us note

$$g(x) = f(x) - \sum_{l=0}^{r-1} a_l(x) \sum_{k=0}^{r-l-1} \Delta^{k+l} y(0) \frac{x^{[k]}}{k!},$$

$$F_i(x) = a_r(x) \psi_i(x) + \sum_{l=0}^{r-1} a_l(x) \psi_{r-l,r-l+i}(x), \quad (7.7)$$

equality (7.6) can be rewritten in a more compact form:

$$\sum_{i=0}^{N-1} y_{r,r+i} F_i(x) = g(x), \quad x \in \Omega_N. \quad (7.8)$$

Thus we came to a system of N linear equations (7.8) with unknown coefficients $y_{r,r+i}$ ($i = 0, 1, \dots, N-1$). This system has a unique solution if functions $\{F_i(x)\}_{i=0}^{N-1}$ are linearly independent on the set Ω_N or (which is the same) the determinant of the matrix $[F_i(x)]_{0 \leq i, x \leq N-1}$ not equals zero. In this case we can find unknown coefficients $y_{r,r+i}$ ($i = 0, 1, \dots, N-1$), solving the system of equations (7.8). The same task can be solved in a different way. Namely, suppose $\{\varphi_k(x)\}_{k=0}^{N-1}$ is a orthonormal system with weight $\rho(x)$, consisting of functions $\varphi_k(x)$, given on the Ω_N . Then we can each of the functions $F_i(x)$ ($i = 0, 1, \dots, N-1$) expand in the finite Fourier series on the system $\{\varphi_k(x)\}_{k=0}^{N-1}$ and get the representation

$$F_i(x) = \sum_{k=0}^{N-1} \hat{F}_{i,k} \varphi_k(x), \quad i, x \in \Omega_N, \quad (7.9)$$

where $\hat{F}_{i,k} = \sum_{j=0}^{N-1} F_i(j)\varphi_k(j)\rho(j)$. Similarly,

$$g(x) = \sum_{k=0}^{N-1} \hat{g}_k \varphi_k(x), \quad x \in \Omega_N, \quad (7.10)$$

where $\hat{g}_k = \sum_{j=0}^{N-1} g(j)\varphi_k(j)\rho(j)$.

Let's insert in (7.8) instead of $F_i(x)$ and $g(x)$ right-hand parts of the equalities (7.9) and (7.10). Then we get

$$\sum_{k=0}^{N-1} \left(\sum_{i=0}^{N-1} y_{r,r+i} \hat{F}_{i,k} \right) \varphi_k(x) = \sum_{k=0}^{N-1} \hat{g}_k \varphi_k(x), \quad x \in \Omega_N. \quad (7.11)$$

From (7.11), we get the system of equations

$$\sum_{i=0}^{N-1} y_{r,r+i} \hat{F}_{i,k} = \hat{g}_k, \quad k \in \Omega_N, \quad (7.12)$$

equivalent to the system (7.8).

There is also a problem of approximate solving of the system (7.12), «dropping» from it coefficients $y_{r,r+i}$, for which $i = n+1, \dots, N-1$ and coefficients $\hat{g}_k$ with $k = m+1, \dots, N-1$, where $n \leq m \leq N-1$. In other words, considered system

$$\sum_{i=0}^n y_{r,r+i} \hat{F}_{i,k} = \hat{g}_k, \quad k = 0, 1, \dots, m, \quad (7.13)$$

where $n \leq m \leq N-1$. We can solve this system, for example, using for example least square method. Which of systems (7.8) or (7.13) for finding the coefficients $y_{r,r+i}$ we should prefer? It depends, in particular, on which character of errors of the values of function $g(x)$, standing in the right-hand part of (7.8). If they are random values with distribution law close to normal, then we can recommend to use the system (7.13), choosing n and m essentially less, then $N-1$. On the other hand, if N is not too big and values of the function $g(x)$ are given with satisfactory accuracy, then we can use the system (7.8).

8. About solving the Cauchy problem for the difference equation using the system $\{\tau_{r,k}^{0,0}(x, N)\}_{k=0}^{N-1+r}$

In this section we briefly dwell on the problem of solving the system of linear equations (7.8) in the particular case when as a source orthonormal system $\{\psi_k(x)\}_{k=0}^{N-1}$ we take the system of Cebyshev polynomials $\{\tau_k^{0,0}(x, N)\}_{k=0}^{N-1}$, orthonormal on Sobolev on the net Ω_N with weight $\mu(x) = \mu(x; 0, 0, N) = 2/N$. Corresponding system of polynomials $\{\tau_{r,k}^{0,0}(x, N)\}_{k=0}^{N-1+r}$, orthonormal on Sobolev on the net Ω_{N+r} , are given with equalities (5.3) – (5.4), from which when $0 \leq l \leq r-1$ we deduce

$$\psi_{r-l,k}(x) = \tau_{r-l,k}^{0,0}(x, N) = \frac{x^{[k]}}{k!}, \quad 0 \leq k \leq r-l-1, \quad (8.1)$$

$$\begin{aligned}\psi_{r-l, r-l+k}(x) &= \tau_{r-l, r-l+k}^{0,0}(x, N) = \\ &= \frac{(-1)^k}{\{h_{k,N}^{0,0}\}^{1/2}} \sum_{j=0}^k \frac{(-1)^j k^{[j]}(k+1)_j x^{[j+r-l]}}{j!^2 (N-1)^{[j]}(j+r-l)^{[r-l]}}, \quad 0 \leq k \leq r-l-1, \\ \psi_{r-l, r-l+k}(x) &= \tau_{r-l, r-l+k}^{0,0}(x, N) = (-1)^{r-l} \left\{ \frac{h_{k-r+l, N-r+l}^{r-l, r-l}}{h_{k,N}^{0,0}} \right\}^{\frac{1}{2}} \times \\ &\times \frac{x^{[r-l]}(N-1+r-l-x)^{[r-l]}}{(N-1)^{[r-l]} k^{[r-l]}} \tau_{k-r+l}^{r-l, r-l}(x-r+l, N-r+l), \quad (8.2)\end{aligned}$$

where $r-l \leq k \leq N-1$. Using equalities (8.1) – (8.2) and (7.7), we can find values of elements of the matrix $[F_i(x)]_{0 \leq i, x \leq N-1}$, if only we will find a way of calculating values of Chebyshev polynomials $\tau_{k-r+l}^{r-l, r-l}(x-r+l, N-r+l)$ ($r-l \leq k \leq N-1$), that appear in (8.2). For this purpose we can turn to the recurrent formula (2.14), in which we change α with $\nu = r-l$, and N with $N-\nu$. As a result we get a formula, which we can rewrite in the next more form, more convenient for numerical calculations

$$\begin{aligned}\tau_{k-\nu}^{\nu, \nu}(x-\nu, N-\nu) &= \hat{\kappa}_{k-\nu}^{\nu, \nu} \left(\frac{2x}{N+\nu-1} - 1 \right) \tau_{k-\nu-1}^{\nu, \nu}(x-\nu, N-\nu) - \\ &\quad \hat{\mu}_{k-\nu}^{\nu, \nu} \tau_{k-\nu-2}^{\nu, \nu}(x-\nu, N-\nu), \quad 2 \leq k-\nu \leq N-\nu-1, \\ \hat{\kappa}_{k-\nu}^{\nu, \nu} &= \left[\frac{N+\nu-1}{N+k} \frac{N+\nu-1}{N-k} \frac{2k-1}{k-\nu} \frac{2k+1}{k+\nu} \right]^{\frac{1}{2}}, \\ \hat{\mu}_{k-\nu}^{\nu, \nu} &= \left[\frac{N+k-1}{N+k} \frac{N-k+1}{N-k} \frac{k-\nu-1}{k-\nu} \frac{k+\nu-1}{k+\nu} \frac{2k+1}{2k-3} \right]^{\frac{1}{2}}, \\ \tau_0^{\nu, \nu}(x-\nu, N-\nu) &= \left[\frac{(2\nu+1)!}{2^{2\nu+1} \nu!^2} \right]^{\frac{1}{2}} = \left[\frac{2\nu+1}{2} \prod_{j=1}^{\nu} \frac{\nu+j}{4j} \right]^{\frac{1}{2}} \equiv \mathcal{A}_{\nu}, \\ \tau_1^{\nu, \nu}(x-\nu, N-\nu) &= \mathcal{A}_{\nu} \left[(2\nu+3) \frac{N+\nu-1}{N-\nu-1} \frac{N+\nu-1}{N+\nu+1} \right]^{\frac{1}{2}} \left(\frac{2x}{N+\nu-1} - 1 \right).\end{aligned}$$

Список литературы

- [1] Iserles A., Koch P.E., Norsett S.P. and Sanz-Serna J.M. On polynomials orthogonal with respect to certain Sobolev inner products // Journal of Approximation Theory, 1991. Vol. 65, Issue 2. P. 151–175.
- [2] Marcellan F., Alfaro M. and Rezola M.L. Orthogonal polynomials on Sobolev spaces: old and new directions // Journal of Computational and Applied Mathematics, 1993. Vol. 48, No 1–2. P. 113–131.
- [3] Meijer H.G. Laguerre polynomials generalized to a certain discrete Sobolev inner product space // Journal of Approximation Theory, 1993. Vol. 73, Issue 1. P. 1–16.
- [4] Kwon K.H. and Littlejohn L.L. The orthogonality of the Laguerre polynomials $\{L_n^{(-k)}(x)\}$ for positive integers k // Ann. Numer. Anal, 1995. Vol. 2. P. 289–303.

- [5] Kwon K.H. and Littlejohn L.L. Sobolev orthogonal polynomials and second-order differential equations // Rocky Mountain J. Math, 1998. Vol. 28, Issue 2. P. 547–594.
- [6] Marcellan F., Xu Y. On Sobolev orthogonal polynomials // Expositiones Mathematicae, 2015. Vol. 33, Issue 3, P. 308–352.
- [7] Sharapudinov I.I. Approximation of discrete functions and Chebyshev polynomials orthogonal on the uniform grid // Mathematical notes, 2000. Vol. 67, Issue 3. P. 389–397.
- [8] Sharapudinov I.I. Approximation of functions of variable smoothness by Fourier-Legendre sums // Sbornik: Mathematics. 2000. Vol. 191, Issue 5. P. 191–759.
- [9] Sharapudinov I.I. Approximation properties of the operators $\mathcal{Y}_{n+2r}(f)$ and of their discrete analogs // Mathematical notes, 2002. Vol. 72, Issue 5. P. 705–732.
- [10] Sharapudinov I.I. Mixed series in ultraspherical polynomials and their approximation properties // Sbornik: Mathematics, 2003. Vol. 194. Issue 3. P. 194–223.
- [11] Sharapudinov I.I. Mixed series by classical orthogonal polynomials // Daghestan Electronic Mathematical Reports, 2015. Issue 3, P. 1–254. (in Russian)
- [12] Sharapudinov I.I. Mixed series on Chebyshev polynomials, orthogonal on the uniform net // Mathematical notes, 2005. Vol. 78, Issue 3. P. 403–423.
- [13] Sharapudinov I.I. Approximation properties of mixed series in terms of Legendre polynomials on the classes W^r // Sbornik: Mathematics, 2006. Vol. 197. Issue 3. P. 433–452.
- [14] Sharapudinov T.I. Approximation properties of mixed series on Chebyshev polynomials, orthogonal on the uniform net // Herald of Daghestan Scientific Center, 2007. Issue 29. P. 12–23. (in Russian)
- [15] Sharapudinov I.I. Approximation properties of the Vallée-Poussin means of partial sums of a mixed series of Legendre polynomials // Mathematical notes, 2008. Vol. 84. Issue 3. P. 417–434.
- [16] Sharapudinov I.I., Muratova G.N. Some Properties of r -fold Integrated Series on Fourier – Haar System // Izv. Sarat. Un-ta Nov. Ser., Ser. Matematika. Mekhanika. Informatika, 2009. Vol. 9. Issue 1. P. 68–76. (in Russian)
- [17] Sharapudinov I.I., Sharapudinov T.I. Mixed series of Jacobi and Chebyshev polynomials and their discretization // Mathematical notes, 2010. Vol. 88. Issue 1. P. 112–139.
- [18] Sharapudinov I.I. Sistemy funktsiy, ortogonalnih po Sobolevu, porozhdenne ortogonalnimi funktsiyami // Sovremennye problemy teorii funktsiy i ih prilozheniya. Materialy 18-y mezhdunarodnoy Saratovskoy zimney shkoly, January 27 – February 3, 2016. P. 329–332. (in Russian)
- [19] Chebyshev P.L. O nepreryvnyh drobyah. (1855), Poln. sobr. soch. (Izd. AN SSSR, M., 1947). Vol. 2, P. 103–126. (in Russian)
- [20] Chebyshev P.L. Ob odnom novom ryade Poln. sobr. soch. (Izd. AN SSSR, M., 1947) Vol. 2, P. 236–238. (in Russian)
- [21] Chebyshev P.L. Ob interpolirovanii po sposobu naimenshih kvadratov (1859), Poln. sobr. soch. (Izd. AN SSSR, M., 1947). Vol. 2, P. 314–334. (in Russian)
- [22] Chebyshev P.L. Ob interpolirovanii (1864), Poln. sobr. soch. (Izd. AN SSSR, M., 1947). Vol. 2, P. 357–374. (in Russian)
- [23] Chebyshev P.L. Ob interpolirovanii velichin ravnotostoyashih (1875), Poln. sobr. soch. (Izd. AN SSSR, M., 1947). Vol. 2, P. 66–87. (in Russian)
- [24] Solodovnikov V.V., Dmitriev A.N., Egupov N.D. Spektralnie metody rascheta i proektirovaniya sistem upravleniya. Moskva: Mashinostroenie, 1986. 440 p. (in Russian)
- [25] Trefethen L.N. Spectral methods in Matlab. SIAM: Philadelphia, 2000. 165 p.
- [26] Trefethen L.N. Finite difference and spectral methods for ordinary and partial differential equation. Cornell University, 1996. 299 p.

- [27] Sharapudinov I.I. Asymptotic properties and weighted estimates for Chebyshev – Hahn orthogonal polynomials // Mathematics of the USSR-Sbornik, 1992. Vol. 72. Issue 2. P. 387–401. (in Russian)
- [28] Sharapudinov I.I. Ob asimptotike mnogochkenov Chebysheva, ortogonalnyh na konechnoy sisteme tochek // Vestnik MGU. Seriya 1, 1992. Issue 1. P. 29–35. (in Russian)
- [29] Sharapudinov I.I. Mnogoshleny, ortogonalnye na diskretnyh setkah. Makhachkala: Izdatelstvo Dag. Gos. Ped. Un-ta, 1997. 234 p. (in Russian)
- [30] Magomed-Kasumov M.G. Priblizhennoe reshenie obyknovennyh differentsialnyh uravneniy s ispolzovaniem smeshannyh ryadov po sisteme Haara // Sovremennye problemy teorii funktsiy i ih prilozheniya. Materialy 18-y mezhdunarodnoy Saratovskoy zimney shkoly, January 27 – February 3, 2016. P. 176–178. (in Russian)

I. I. Sharapudinov (И. И. Шарапудинов)

Daghestan scientific center of RAS, Daghestan State
Pedagogical University, Southern Mathematical Institute of
the Vladikavkaz Scientific Center of the RAS
E-mail: sharapud@mail.ru

Поступила в редакцию
07.04.2016

T. I. Sharapudinov (Т. И. Шарапудинов)

Daghestan scientific center of RAS, Southern Mathematical
Institute of the Vladikavkaz Scientific Center of the RAS
E-mail: sharapudinov@gmail.com