

UDC 517.5

Romi F. Shamoyan

On some new sharp embedding theorems in area Nevanlinna spaces and related problems

We provide some new sharp embedding theorems for analytic area Nevanlinna spaces in the unit disk extending some previously known assertions in various directions.

Bibliography: 17 items.

Мы приводим некоторые новые точные теоремы вложения пространств Неванлинны с аналитической площадью в единичный круг, расширяющие некоторые ранее известные утверждения в различных направлениях.

Библиография: 17 названий.

Keywords: area Nevanlinna spaces, embedding theorems, Nevanlinna characteristic.

Ключевые слова: площадь пространства Неванлинны, теоремы вложения, характеристика Неванлинны.

1. Introduction

Let Ω be a bounded pseudoconvex domain in \mathbb{C}^n . Let also T_Ω be a tubular domain over symmetric cone in \mathbb{C}^n . Let further $\mu(\tilde{\mu})$ be positive Borel measure on T_Ω (on Ω). Let also Δ^α , $\alpha \geq 0$, be determinant function on T_Ω , and $\delta^\alpha(w) = \text{dist}^\alpha(w, \partial\Omega)$, $\alpha > -1$. Let dv (dV) be the normalized Lebesgues measure on T_Ω (on Ω). $B(z, r)$ ($\tilde{B}(z, r)$) $z \in \Omega$ ($z \in T_\Omega$) be Bergman or Kobayashi ball in tubular domains over symmetric cones or pseudoconvex bounded domains in \mathbb{C}^n , (see [1], [2]). Let N_α^p be the classical area Nevanlinna space of T_Ω (on Ω). Then we can state the following based on recent techniques developed in recent papers [1], [2].

Namely it is easy to show that if for some $\tilde{\alpha}$, $\tilde{\alpha} > 0$,

$$\mu(B(z, r)) \leq c_0(\Delta^{\tilde{\alpha}}(Imz)), \quad r > 0, \quad z \in T_\Omega.$$

Then

$$\begin{aligned} \int_{T_\Omega} (\log^+ |f(z)|)^p d\mu(z) &\leq c \|f\|_{N_\alpha^g(T_\Omega)}^p \\ &= c \int_{T_\Omega} (\log^+ |f(w)|)^p (\Delta^\alpha(Imw)) dv(z), \end{aligned}$$

for $\alpha > -1$ and for $1 \leq p < \infty$, and also

$$\int_{\Omega} (\log^+ |f(z)|)^p d\tilde{\mu}(z) \leq \tilde{c} \|f\|_{N_\alpha^p(\Omega)}^p = \tilde{c} \int_{\Omega} (\log^+ |f(w)|)^p \delta^\alpha(w) dV(w)$$

for $1 \leq p < \infty$, $\alpha > -1$. If the following condition holds $\tilde{\mu}(\tilde{B}(z, r)) \leq c_2(\delta^{\tilde{\alpha}}(z))$, $z \in \Omega$, $r > 0$, $\alpha > 0$ for some $\tilde{\alpha}$ and for some positive constants c , c_0 , \tilde{c} , c_2 . Same type results can be provided in the polydisk and in bounded symmetric domains based on the same known technique (see, for example, [17] for unit ball case). These type of conditions on measure will be called Carleson type conditions.

Throughout the paper, we write C or c (with or without lower indexes) to denote a positive constant which might be different at each occurrence (even in a chain of inequalities), but is independent of the functions or variables being discussed.

Let w be a function from a set S of all positive growing functions, $w \in L^1(0, 1)$ such that there are two numbers $m_w > 0$, $M_w > 0$ and a number $q_w \in (0, 1)$ such that

$$m_w \leq \frac{w(\lambda\tau)}{w(\tau)} < M_w, \quad \tau \in (0, 1), \quad \lambda \in [q_w, 1],$$

(this condition will be used in proofs of $2^\circ \Rightarrow 1^\circ$ in theorems 1 and 2). Let $w \in S$, then there are measurable functions $\varepsilon(x)$, $q(x)$ so that

$$w(x) = \exp \left\{ q(x) + \int_x^1 \frac{\varepsilon(u)}{u} du \right\}, \quad x \in (0, 1).$$

This characterization gives various examples of functions from S class. See properties of these classes in [15].

We need the following simple estimate for proofs of theorems 1-3, (see [6]):

Let $s > 1$, $w \in S$ then

$$\int_0^1 \frac{w(1-r)}{(1-\rho r)^s} dr \leq \frac{Cw(1-\rho)}{(1-\rho)^{s-1}}, \quad \rho \in (0, 1),$$

(this condition will be used in proofs of $1^\circ \Rightarrow 2^\circ$ in theorems 1 and 2).

Let $T(r, f)$ be, as usual, Nevanlinna characteristic of analytic function f

$$T(r, f) = \int_T \log^+ |f(r\xi)| d\xi, \quad r \in (0, 1).$$

Let us define $\Delta_{k,s}$ as a standard dyadic cube in the unit disk (see [8])

$$\Delta_{k,s} = \left\{ z \in \mathcal{D} : 1 - \frac{1}{2^k} \leq |z| < 1 - \frac{1}{2^{k+1}}, \frac{2\pi s}{2^{k+1}} \leq \arg z < \frac{2\pi(s+1)}{2^{k+1}} \right\},$$

$$s = -2^{k+1}, \dots, 2^{k+1} - 1, \quad k = 0, 1, 2, \dots$$

Let further $|\Delta_{k,s}|$ be Lebesgue measure of $\Delta_{k,s}$.

In the unit disk \mathcal{D} , $T = \partial\mathcal{D} = \{|z| = 1\}$ the following sharp results were provided recently in [6].

THEOREM A. *Let μ be finite nonnegative Borel measure defined on subsets of \mathcal{D} . Let $1 \leq p < \infty$. Then the following are equivalent:*

1. $\int_{\mathcal{D}} (\ln^+ |f(\xi)|)^p d\mu(\xi) \leq c_3 \int_0^1 w(1-r) T^p(r, f) dr < \infty,$
2. $\mu(\Delta_l(\theta)) \leq c_4 w(l) l^{p+1}, \quad \theta \in [-\pi, \pi], \quad l \in (0, 1),$

$$\Delta_l(\theta) = \{z \in \mathcal{D} : (1-l) < |z| < 1, \quad |\arg z - \theta| \leq \frac{l}{2}\}.$$

THEOREM B. *Let μ be a finite nonnegative Borel measure defined on subset of \mathcal{D} . Let $0 < p < 1$, $r_k = 1 - \frac{1}{2^k}$, $k = 0, 1, 2, \dots$, Then the following are equivalent.*

1. $\int_{\mathcal{D}} (\ln^+ |f(\xi)|)^p d\mu(\xi) \leq c_5 \int_0^1 w(1-r) T^p(r, f) dr < +\infty.$
2. $\sum_{s=-2^k}^{2^k-1} \mu(\Delta_{k,s})^{\frac{1}{1-p}} \leq c_6 (1-r_k)^{\frac{1+p}{1-p}} \left(w(1-r_k)^{\frac{1}{1-p}} \right).$

Various basic properties of Nevanlinna type spaces can be seen in [4]. Results of this paper can be extended partially to more general area Nevanlinna type spaces studied in [9].

We use heavily some nice technique developed in [6] to extend the sharp results.

We refer to [3] for similar type results concerning sharp embeddings in area Nevanlinna type spaces in the unit disk. See also [15] for various new embeddings in area Nevanlinna spaces in the unit disk.

2. Main results

The goal of this note to extend those sharp results in theorems A and B using similar ideas to other values of parameters. Namely we obtained the following sharp embedding theorems for area Nevanlinna type spaces in the unit disk \mathcal{D} .

THEOREM 1. *Let $q \leq p$, $p > 1$. Let μ be positive Borel measure on \mathcal{D} . Then the following are equivalent:*

1. $\int_{\mathcal{D}} (\log^+ |f(z)|)^p d\mu(z) \leq c_8 \int_0^1 w(1-r) \left(\int_{-\pi}^{\pi} \log^+ |f(r\xi)| d\xi \right)^q dr.$
2. $\mu(\Delta_l) \leq c_7 w(l) \frac{p}{q} l^{\frac{p}{q}+p}.$

Note obviously for $q = p$ we obtain immediately Theorem A.

The following result is another sharp extension of Theorem A.

THEOREM 2. *Let $q \leq 1$ and let μ be positive Borel measure on \mathcal{D} . Then the following conditions are equivalent:*

1. $\int_{\mathcal{D}} \log^+ |f(z)| d\mu(z) \leq c_9 \int_0^1 w(1-r) \left(\int_{-\pi}^{\pi} \log^+ |f(r\xi)| d\xi \right)^q dr.$
2. $\mu(\Delta_l) \leq c_{10} w(l) \frac{1}{q} l^{\frac{1}{q}+1}.$

The following sharp result is a direct extension of Theorem B.

We have the following result.

THEOREM 3. *Let μ be positive Borel measure on \mathcal{D} . Let $q \leq p$, $p < 1$. Then the following conditions are equivalent:*

$$\int_{\mathcal{D}} (\log^+ |f(z)|)^p d\mu(z) \leq C_2 \int_0^1 \left(\int_T \log^+ |f(r\xi)| d\xi \right)^q w(1-r) dr. \quad (1)$$

$$\left(\sum_{s=-2^k}^{2^{k-1}} \mu(\Delta_{k,s})^{\frac{1}{1-p}} \right)^{1-p} \leq C_1 w(1-r_k)^{\frac{p}{q}} (1-r_k)^{\frac{(q+1)p}{q}} \quad (2)$$

REMARK 1. Note (2) type condition, namely 2° from Theorem B, with $p = q$ is also sufficient for embedding of the type

$$\int_{\mathcal{D}} (\log^+ |f(z)|)^{\tilde{p}} d\mu(z) \leq C \int_0^1 \left(\int_T \log^+ |f(r\xi)| d\xi \right)^q w(1-r) dr$$

where $q \leq 1$, $\tilde{p} > 1$, $q \leq \tilde{p}$. This can be seen from our proof.

REMARK 2. Similar sharp theorems with very similar proof can be obtained if we replace the right side in Theorem 1 and Theorem 2 by spaces with quasi norms

$$\sum_{k \geq 0} \left(\int_{1-2^k}^{1-2^{-(k+1)}} w(1-r) \left(\int_T \log^+ |f(r\xi)| d\xi \right)^q dr \right)^s,$$

$$\int_0^1 \left(\int_{|z| \leq r} \log^+ |f(w)| (1-|w|)^\alpha dm_2(w) \right)^q w(1-r) dr,$$

(readers can easily recover such theorems based on our proofs below) for $0 < s \leq 1$, $\alpha > -1$, $0 < q < \infty$, where dm_2 is a Lebesgues measure in \mathcal{D} , where w is a weight from a S function class, (see [6] for these weights) with some additional restrictions on parameters.

References

- [1] M. Abate, J. Raissy, A. Saracco. Toeplitz operators and Carleson measures in strongly pseudoconvex domains // Journal of Functional Analysis. 2012. Vol. 263. Issue 11. pp. 3449–3491.
- [2] D. Bekolle, A. Bonami, G. Garrigos, C. Nana, M. M. Peloso, F. Ricci. Lecture notes on Bergman projectors in tube domains over cones: an analytic and geometric viewpoint. Proceeding of the International Workshop on Classical Analysis. Yaounde. 2001. 75 pages.

-
- [3] B. R. Choe, H. Koo and W. Smith. Carleson measures for the area Nevanlinna spaces and applications // *Journal d'Analyse Mathématique*. 2008. Vol. 104. Issue 207. <https://doi.org/10.1007/s11854-008-0022-8>.
 - [4] H. Hedenmalm, B. Korenblum and K. Zhu. *Theory of Bergman spaces* // *Graduate Texts in Mathematics*. Springer-Verlag, New York. 2000. Vol. 199.
 - [5] J. Ortega, J. Fabrega. Mixed norm spaces and interpolation // *Studia Mathematica*. 1994. Vol. 109. Issue 3. pp. 233–254.
 - [6] F. Shamoyan, E. Rodikova. L^p estimates in analytic function spaces with restriction on Nevanlinna characteristic // *Vestnik Bryansk Gos. Univ.* 2012. Vol. 4. pp. 80–86. (in Russian)
 - [7] R. Shamoyan. On some characterizations of Carleson type measure in the unit ball // *Banach Journal of Mathematical Analysis*. 2009. Vol. 3. Issue 2. pp. 42–48.
 - [8] R. Shamoyan. On multipliers from spaces of the Bergman type to the Hardy spaces in a polydisk // *Ukrainian Mathematical Journal*. 2000. Vol. 52. pp. 1606–1617.
 - [9] R. Shamoyan, H. Li. Descriptions of zero sets and parametric representations of certain analytic area Nevanlinna type classes in the unit disk // *Kragujevac Journal of Mathematics*. 2010. Vol. 34. pp. 73–89.
 - [10] R. Shamoyan, H. Li. Some remarks on the action of Lusin area operator in Bergman spaces of the unit ball // *Acta Universitatis Apulensis*. 2012. Vol. 29. pp. 31–45.
 - [11] R. Shamoyan, S. Maksakov. Embedding theorems for weighted anisotropic spaces of holomorphic functions in strongly pseudoconvex domains // *ROMAI J.* 2017. Vol. 13. Issue 1. pp. 71–92.
 - [12] R. Shamoyan, O. Mihić. On some properties of holomorphic spaces based on Bergman metric ball and Luzin area operator // *The Journal of Nonlinear Sciences and Applications*. 2009. Vol. 2. Issue 3. pp. 183–194.
 - [13] R. Shamoyan, O. Mihić. On some new sharp embedding theorems in minimal and pseudoconvex domains // *Czechoslovak Mathematical Journal*. 2016. Vol. 66. Issue 2. pp. 527–546.
 - [14] R. Shamoyan, O. Mihić. On some new sharp estimates in analytic

- Herz-type function spaces in tubular domains over symmetric cones // Czechoslovak Mathematical Journal. 2018. Vol. 68. pp. 1033–1050.
- [15] R. Shamoyan, O. Mihić. On zero sets and embeddings of some new analytic function spaces in the unit disk // Kragujevac Journal of Mathematics. 2014. Vol. 38. Issue 2. pp. 229–244.
- [16] R. Shamoyan, O. Mihić. Embedding theorems for weighted anisotropic spaces of holomorphic functions in tubular domains // ROMAI J. 2017. Vol. 13. Issue 1. pp. 93–115.
- [17] K. Zhu. Spaces of holomorphic functions in the unit ball. Springer-Verlag, New York. 2005.

Romi F. Shamoyan (Роми Ф. Шамоян)

Department of Mathematics, Bryansk State Technical University, Bryansk 241050, Russia

E-mail: rshamoyan@gmail.com

Received

25.08.2020

Final version

07.07.2021