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On Solvability of an Initial Value Problem for Hilfer type Fractional Differential Equation with Nonlinear Maxima

In this article we consider the questions of one-valued solvability of initial value problem for a nonlinear Hilfer type fractional differential equation with nonlinear maxima. By the aid of uncomplicated integral transformation based on Dirichlet formula, this initial value problem is reduced to the nonlinear Volterra type fractional integral equation with nonlinear maxima. It is proved the theorem of existence and uniqueness of the solution of given initial value problem in an interval under consideration. It is proved also the stability of the desired solution with respect to given parameter.

Bibliography: 26 items.

В статье рассматриваются вопросы однозначной разрешимости начальной задачи для нелинейного дифференциального уравнения типа Хильфера дробного порядка с нелинейными максимумами. С помощью несложного интегрального преобразования, основанного на формуле Дирихле, эта начальная задача сводится к нелинейному интегральному уравнению типа Вольтерра дробного порядка с нелинейными максимумами. Доказана теорема существования и единственности решения заданной начальной задачи на рассматриваемом интервале. Доказана также устойчивость искомого решения по заданному параметру.

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Ключевые слова: Обыкновенное дифференциальное уравнение, начальная задача, нелинейные максимумы, оператор Хильфера, однозначная разрешимость.

1. Introduction

Let $\big(t_0;\, b\big) \subset \mathbb{R}^+ \equiv [0;\, \infty)$ be a finite interval on the set of positive real numbers, and let $\alpha > 0$. The Riemann–Liouville α-order fractional integral of a function $\eta(t)$ is defined as follows:

$$I^{\alpha}_{t_0+}\eta(t) = \frac{1}{\Gamma(\alpha)} \int\limits_{t_0}^{t} (t-s)^{\alpha-1}\eta(s)\, d\, s, \quad \alpha > 0, \quad t \in \big(t_0;\, b\big),$$

where $\Gamma(\alpha)$ is the Gamma function [1, p. 112].

Let $n-1 < \alpha \le n$, $n \in \mathbb{N}$. The Riemann–Liouville α-order fractional derivative of a function $\eta(t)$ is defined as follows [2, Vol. 1, p. 27]:

$$D^{\alpha}_{t_0+}\eta(t) = \frac{d^n}{dt^n} I^{n-\alpha}_{t_0+}\eta(t), \quad t \in \big(t_0;\, b\big).$$

The Caputo α-order fractional derivative of a function $\eta(t)$ is defined [2, Vol. 1, p. 34] by

$${}_*D^{\alpha}_{t_0+}\eta(t) = I^{n-\alpha}_{t_0+}\eta^{(n)}(t) = \frac{1}{\Gamma(n-\alpha)} \int\limits_{t_0}^{t} \frac{\eta^{(n)}(s)\, d\, s}{(t-s)^{\alpha-n+1}}, \quad t \in \big(t_0;\, b\big).$$

Both the derivatives are reduced to the n-th order derivatives for $\alpha = n \in \mathbb{N}$ [2, Vol. 1, pp. 27, 34]:

$$D^{n}_{t_0+}\eta(t) = {}_*D^{n}_{t_0+}\eta(t) = \frac{d^n}{dt^n}\eta(t), \quad t \in \big(t_0;\, b\big).$$

The so-called generalized Riemann–Liouville fractional derivative (referred to as the Hilfer fractional derivative) of order α, $n-1 < \alpha \le n$, $n \in \mathbb{N}$ and type β, $0 \le \beta \le 1$, is defined by the following composition of three operators: [1, p. 113]:

$$D^{\alpha,\,\beta}_{t_0+}\eta(t) = I^{\beta(n-\alpha)}_{t_0+} \frac{d^n}{dt^n} I^{(1-\beta)(n-\alpha)}_{t_0+}\eta(t), \quad t \in \big(t_0;\, b\big).$$

For $\beta = 0$, this operator is reduced to the Riemann–Liouville fractional derivative $D^{\alpha,0}_{t_0+} = D^{\alpha}_{t_0+}$ and the case $\beta = 1$ corresponds to the Caputo fractional derivative $D^{\alpha,1}_{t_0+} = {}_*D^{\alpha}_{t_0+}$.

Let $\gamma = \alpha + \beta n - \alpha \beta$. It is easy to see that $\alpha \leq \gamma \leq n$. Then it is convenient to use another designation for the operator $D^{\alpha,\beta}_{t_0+}\eta(t)$:

$$D^{\alpha,\gamma}\eta(t) = D^{\alpha,\beta}_{t_0+}\eta(t).$$

The generalized Riemann–Liouville operator was introduced in [1] by R. Hilfer on the basis of fractional time evolutions that arise during the transition from the microscopic scale to the macroscopic time scale. Using the integral transforms, he investigated the Cauchy problem for the generalized diffusion equation, the solution of which is presented in the form of the Fox H-function. Note [3, 4], where the generalized Riemann–Liouville operator was used in studying dielectric relaxation in glass-forming liquids with different chemical compositions. In [5], the properties of the generalized Riemann–Liouville operator were investigated in a special functional space, and an operational method was developed for solving fractional differential equations with this operator. Based on the results of the work [5], the authors of [6] have developed an operational method for solving fractional differential equations containing a finite linear combination of the generalized Riemann–Liouville operators with various parameters.

Fractional calculus plays an important role for the mathematical modeling in many scientific and engineering disciplines [7]. In [8] it is considered problems of continuum and statistical mechanics. In [9] is studied the mathematical problems of Ebola epidemic model. In [10] and [11] are studied the fractional model for the dynamics of tuberculosis infection and novel coronavirus (nCoV-2019), respectively. The construction of various models of theoretical physics by the aid of fractional calculus is described in [2, Vol. 4, 5], [12, 13]. A specific interpretation of the Hilfer fractional derivative, describing the random motion of a particle moving on the real line at Poisson paced times with finite velocity is given in [14]. A detailed review of the application of fractional calculus in solving problems of applied sciences is given in [2, Vol. 6-8], [15]. More detailed information related to the theory of fractional integro-differentiation, including the Hilfer fractional derivative, one can find in [16]. In [17] by analytical method is studied the unique solvability of boundary value problem for weak nonlinear partial differential equations of mixed type with fractional Hilfer operator. In [18] is studied the solvability of nonlocal problem for a mixed type fourth-order differential equation with Hilfer fractional operator. In [19] it is considered an inverse problem for a mixed type integro-differential equation with fractional order Caputo operators.

In the present paper we consider the questions of one-valued solvability and stability of solution for a Hilfer type fractional differential equation with nonlinear maxima. This equation we solve under initial value condition. Differential equations with maxima play an important role in solving control problems in the sale of goods and investment of manufacturing companies in a market economy [20]. In [21], it is justified that the theoretical study of differential equations with maxima is relevant.

2. Formulation of the problem and fractional integral equation

We consider the Hilfer type fractional differential equation on an interval $(t_0;\, T)$:

$$D^{\alpha,\gamma}x\,(t)+\omega x\,(t)=f\left(t,\,x\,(t),\,\max\left\{x\,(\theta)\,|\,\theta\in[q_1(t);\,q_2(t)]\right\}\right) \qquad (1)$$

under initial value condition

$$\lim_{t\to t_0} J_{t_0+}^{1-\gamma}x(t)=x_0,\quad x(t)=\varphi(t),\quad t\notin(t_0;\,T)\,, \qquad (2)$$

where $f\,(t,\,u,\,\vartheta)\in C\,([t_0;\,T]\times\mathbb{X}\times\mathbb{X})$, $\varphi\,(t)\in C\,\left([0;\,t_0]\cup[T;\,\infty)\right)$, $0<\omega$ is real parameter, $x_0=\mathrm{const}$, $0\le t_0$, $\mathbb{X}\subset\mathbb{R}\equiv(-\infty;\,\infty)$, $q_i=q_i(t,\,x\,(t))\in C\,\left([t_0;\,T]\times\mathbb{X}\right)$, $i=1,2$, $\mathbb{X}$ is closed set. Here $D^{\alpha,\gamma}=J_{t_0+}^{\gamma-\alpha}\frac{d}{dt}J_{t_0+}^{1-\gamma}$, $0<\alpha\le\gamma\le1$ is Hilfer operator and J_{0+}^{ν} is the Riemann–Liouville integral operator, which is defined by the formula

$$J_{t_0+}^{\alpha}\eta(t)=\frac{1}{\Gamma(\alpha)}\int\limits_{t_0}^{t}\frac{\eta(s)\,d\,s}{(t-s)^{1-\alpha}},\quad \alpha>0.$$

We set $0<q_1<q_2<\infty$ and understand it that there possible the cases: 1) $0\le q_1<q_2<t$; 2) $0\le q_1<t$, $t\le q_2<\infty$; 3) $t\le q_1<q_2<\infty$.

Lemma 1. *The solution of the differential equation* (1) *with initial value condition* (2) *on the interval* $(t_0; T)$ *is represented as follows*

$$x(t) = x_0(t-t_0)^{\gamma-1}E_{\alpha,\gamma}\left(-\omega\,(t-t_0)^{\alpha}\right) + \int\limits_{t_0}^{t}(t-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\omega\,(t-s)^{\alpha}\right)\times$$
$$\times f\left(s,\, x(s),\, \max\left\{x(\theta)\,|\,\theta\in[q_1(s,\,x(s));\, q_2(s,\,x(s))]\right\}\right)d\,s, \quad (3)$$

where $E_{\alpha,\gamma}(z)$ *is Mittag–Leffler function and has the form [2, vol. 1, 269–295]*

$$E_{\alpha,\gamma}(z) = \sum_{k=0}^{\infty}\frac{z^{\,k}}{\Gamma(\alpha\,k+\gamma)}, \quad z,\,\alpha,\,\gamma\in(0;\,\infty).$$

Proof. We rewrite the differential equation (1) in the form

$$J^{\gamma-\alpha}_{t_0+}D^{\gamma}_{t_0+}\,x(t) = -\omega\,x(t) + f(t,\,\cdot),$$

where $f(t,\,\cdot) = f\left(s,\,x(s),\,\max\left\{x(\theta)\,|\,\theta\in[q_1(s,\,x(s));\,q_2(s,\,x(s))]\right\}\right)$.

Applying the operator $J^{\alpha}_{t_0+}$ to both sides of this equation and taking into account the linearity of this operator and the formula [6]

$$J^{\gamma}_{t_0+}D^{\gamma}_{t_0+}x(t) = x(t) - \frac{1}{\Gamma(\gamma)}\,J^{1-\gamma}_{t_0+}x(t)|_{\,t=t_0+}(t-t_0)^{\gamma-1},$$

we obtain

$$x(t) = \frac{x_0}{\Gamma(\gamma)}\,(t-t_0)^{\gamma-1} + J^{\alpha}_{t_0+}f(t,\,\cdot) - \omega\,J^{\alpha}_{t_0+}x(t). \quad (4)$$

Using the lemma from [22], we represent the solution of equation (4) in the form

$$x(t) = \frac{x_0}{\Gamma(\gamma)}\,(t-t_0)^{\gamma-1} + J^{\alpha}_{t_0+}f(t,\,\cdot) -$$

$$-\omega\int\limits_{t_0}^{t}(t-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\omega\,(t-s)^{\alpha}\right)\left[\frac{x_0}{\Gamma(\gamma)}\,(s-t_0)^{\gamma-1} + J^{\alpha}_{t_0+}f(s,\,\cdot)\right]d\,s. \quad (5)$$

We rewrite the representation (5) as the sum of two expressions:

$$I_1(t) = x_0 \left[\frac{(t-t_0)^{\gamma-1}}{\Gamma(\gamma)} - \frac{\omega}{\Gamma(\gamma)} \int\limits_{t_0}^{t} \frac{E_{\alpha,\alpha}\left(-\omega\,(t-s)^{\alpha}\right)}{(t-s)^{1-\alpha}} (s-t_0)^{\gamma-1}\,d\,s \right], \quad (6)$$

$$I_2(t) = J^{\alpha}_{t_0+} f\,(t,\,\cdot) - \omega \int\limits_{t_0}^{t} (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\omega\,(t-s)^{\alpha}\right) J^{\alpha}_{t_0+} f\,(s,\,\cdot)\,d\,s. \quad (7)$$

We apply the following representations [2, vol. 1, 269–295]

$$E_{\alpha,\gamma}(z) = \frac{1}{\Gamma(\gamma)} + z\,E_{\alpha,\gamma+\alpha}(z), \quad \alpha > 0, \quad \gamma > 0, \quad (8)$$

$$\frac{1}{\Gamma(k)} \int\limits_{t_0}^{z} (z-t)^{k-1} E_{\alpha,\gamma}\left(-\omega\,t^{\alpha}\right)\,t^{\gamma-1} d\,t =$$
$$= z^{\gamma+k-1} E_{\alpha,\gamma+k}\left(-\omega\,z^{\alpha}\right), \ k > 0, \ \gamma > 0. \quad (9)$$

Then for the integral (6) we obtain representation

$$I_1(t) = x_0 (t-t_0)^{\gamma-1} E_{\alpha,\gamma}\left(-\omega\,(t-t_0)^{\alpha}\right). \quad (10)$$

The integral in (7) is easily transformed to the form

$$\int\limits_{t_0}^{t} (t-\xi)^{\alpha-1} E_{\alpha,\alpha}\left(-\omega\,(t-\xi)^{\alpha}\right) J^{\alpha}_{t_0+} f\,(\xi,\,\cdot)\,d\,\xi =$$
$$= \frac{1}{\Gamma(\alpha)} \int\limits_{t_0}^{t} (t-\xi)^{\alpha-1} E_{\alpha,\alpha}\left(-\omega\,(t-\xi)^{\alpha}\right)\,d\,\xi \int\limits_{t_0}^{\xi} (\xi-s)^{\alpha-1} f\,(s,\,\cdot)\,d\,s =$$
$$= \frac{1}{\Gamma(\alpha)} \int\limits_{t_0}^{t} f\,(s,\,\cdot)\,d\,s \int\limits_{s}^{t} (t-\xi)^{\alpha-1} (\xi-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\omega\,(t-\xi)^{\alpha}\right)\,d\,\xi. \quad (11)$$

Taking the (9) into account the second integral in the last equality of (11) can be written as

$$\int_s^t (t-\xi)^{\alpha-1}(\xi-s)^{\alpha-1}E_{\alpha,\,\alpha}\left(-\omega\,(t-\xi)^{\alpha}\right)\,d\,\xi =$$

$$= \Gamma(\alpha)\,(t-\xi)^{2\,\alpha-1}E_{\alpha,\,2\alpha}\left(-\omega\,(t-\xi)^{\alpha}\right).$$

Then, taking into account (8), we represent (7) in the following form

$$I_2(t) = \int_{t_0}^t (t-\xi)^{\alpha-1}E_{\alpha,\,\alpha}\left(-\omega\,(t-\xi)^{\alpha}\right)\,f\,(\xi,\,\cdot)\,d\,\xi. \tag{12}$$

Substituting (10) and (12) into the sum $x(t) = I_1(t) + I_2(t)$, we obtain (3). The lemma 1 is proved. □

3. Unique solvability of problem (1), (2)

Instead of the integral equation (3) we study the questions of unique solvability of the following integral equation

$$x\,(t,\,\omega)(t-t_0)^{1-\gamma} = \mathcal{I}\,(t;\,x) \equiv x_0 E_{\alpha,\,\gamma}\left(-\omega\left(t-t_0\right)^{\alpha}\right) +$$
$$+ \int_{t_0}^t (t-s)^{\alpha-1}(t-t_0)^{1-\gamma}E_{\alpha,\,\alpha}\left(-\omega\,(t-s)^{\alpha}\right)\,\times$$
$$\times\, f\,(s,\,x\,(s,\,\omega),\,\max\left\{x\,(\theta,\,\omega)\,|\,\theta\in[q_1(s,\,x\,(s,\,\omega));\,q_2(s,\,x\,(s,\,\omega))]\right\})\,d\,s, \tag{13}$$

which has no singularities at the point $t = t_0$.

THEOREM 1. *Let the following three conditions be satisfied:*

1. $\max\left\{\max\limits_{t\notin(t_0;T)}|\,\varphi\,(t)\,|\,;\;\max\limits_{t_0\le t\le T}|\,f\,(t,\,x,\,y)\,|\right\} \le M_0 = \text{const} < \infty$*;*
2. $|f\,(t,\,x_1,\,y_1) - f\,(t,\,x_2,\,y_2)| \le L_0\,(|x_1-x_2| + |y_1-y_2|)$*,* $0 < L_0 = \text{const} < \infty$*;*

3. $|\, q_i(t,\, x_1) - q_i(t,\, x_2)\,| \le L_{0\,i}\,|\, x_1 - x_2\,|$, $0 < L_{0\,i} = \mathrm{const} < \infty$, $i = 1,\, 2$.

Then there exists a unique solution of the initial value problem (1), (2) *in the space of continuous functions* $C\,(t_0;\, T)$, *which can be found by the method of successive approximations:*

$$\begin{cases} x_0(t,\,\omega)(t-t_0)^{1-\gamma} = G(t), \\ x_{\,k+1}\,(t,\,\omega)(t-t_0)^{1-\gamma} = \mathcal{I}\,(t;\, x_{\,k}), \quad k = 0\,,\, 1\,,\, 2\,,\, \ldots, \end{cases} \tag{14}$$

where $G(t) = x_0 E_{\alpha,\,\gamma}\,(-\omega\,(t-t_0)^{\alpha})$.

Proof. Mittag–Leffler function $E_{\alpha,\,\gamma}(z)$ has the following property [23]: we assume that $0 < \alpha < 2$, γ is real constant and $\arg z = \pi$. Then there holds

$$|E_{\alpha,\,\gamma}(z)| \le \frac{A}{1+|z|},$$

where A is positive constant and does not dependent on z. Then it is not difficult to see that from the approximations (14) we obtain that there following estimate holds

$$\left|\,(t-t_0)^{1-\gamma} x_0(t,\,\omega)\,\right| \le |\,x_0|\cdot|\,E_{\alpha,\,\gamma}\,(-\omega\,(t-t_0)^{\alpha})\,| \le |\,x_0|\cdot C_{\,0}, \tag{15}$$

where $|\,E_{\alpha,\,\alpha}\,(-\omega\,(t-s)^{\alpha})\,| \le C_{\,0}$.

By virtue of first condition of the theorem and estimate (15), from successive approximations (14) we obtain

$$|x_1(t,\omega) - x_0(t,\omega)| \le \int\limits_{t_0}^{t} \left|(t-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\omega(t-s)^{\alpha}\right) \times\right.$$
$$\times f\left(s, x_0(s,\omega), \max\left\{x_0(\theta,\omega) | \theta \in \left[q_1\left(s, x_0(s,\omega)\right); q_2\left(s, x_0(s,\omega)\right)\right]\right\}\right)\big|\, ds \le$$
$$\le M_0 \cdot C_0 |x_0| \left|\int\limits_{t_0}^{t} (t-s)^{\alpha-1} ds\right| \le \frac{|x_0|}{\alpha} M_0 \cdot C_0 \cdot (t-t_0)^{\alpha}. \tag{16}$$

We continue the Picard iteration process for the integral equation (13) according to the successive approximations (14). Then, by virtue of conditions of the theorem and taking the estimate (16) into account, we derive

$$
\begin{gathered}
|x_2(t,\omega)-x_1(t,\omega)| \le \int\limits_{t_0}^{t} \left|(t-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\omega(t-s)^{\alpha}\right)\times\right. \\
\times\left[f\left(s, x_1(s,\omega), \max\left\{x_1(\theta,\omega)|\theta\in\left[q_1\left(s,x_1(s,\omega)\right); q_2\left(s,x_1(s,\omega)\right)\right]\right\}\right)-\right. \\
\left.\left.-f\left(s, x_0(s,\omega), \max\left\{x_0(\theta,\omega)|\theta\in\left[q_1\left(s,x_0(s,\omega)\right); q_2\left(s,x_0(s,\omega)\right)\right]\right\}\right)\right]\right| ds \le \\
\le L_0 \int\limits_{t_0}^{t} \left|(t-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\omega(t-s)^{\alpha}\right)\right| \left[|x_1(s,\omega)-x_0(s,\omega)|+\right. \\
+\left|\max\left\{x_1(\theta,\omega)|\theta\in\left[q_1\left(s,x_1(s,\omega)\right); q_2\left(s,x_1(s,\omega)\right)\right]\right\}-\right. \\
\left.-\max\left\{x_0(\theta,\omega)|\theta\in\left[q_1\left(s,x_1(s,\omega)\right); q_2\left(s,x_1(s,\omega)\right)\right]\right\}\right|+ \\
+\left|\max\left\{x_0(\theta,\omega)|\theta\in\left[q_1\left(s,x_1(s,\omega)\right); q_2\left(s,x_1(s,\omega)\right)\right]\right\}-\right. \\
\left.\left.-\max\left\{x_0(\theta,\omega)|\theta\in\left[q_1\left(s,x_0(s,\omega)\right); q_2\left(s,x_0(s,\omega)\right)\right]\right\}\right|\right] ds \le \\
\le L_0 \int\limits_{t_0}^{t} \left|(t-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\omega(t-s)^{\alpha}\right)\right| \left[2\left|x_1(s,\omega)-x_0(s,\omega)\right|+\right. \\
+M_0\left(\left|q_1\left(s,x_1(s,\omega)\right)-\right.\right. \\
\left.\left.\left.-q_1\left(s,x_0(s,\omega)\right)\right|+\left|q_2\left(s,x_1(s,\omega)\right)-q_2\left(s,x_0(s,\omega)\right)\right|\right)\right] ds \le \\
\le \left(2+M_0\left(L_{01}+L_{02}\right)\right) C_0 L_0 \int\limits_{t_0}^{t} (t-s)^{\alpha-1} |x_1(s,\omega)-x_0(s,\omega)| ds \le \\
\le \frac{|x_0|}{\alpha}\left(2+M_0\left(L_{01}+L_{02}\right)\right) M_0 \cdot C_0^2 L_0 \int\limits_{t_0}^{t} (t-s)^{\alpha-1}(s-t_0)^{\alpha} d\,s.
\end{gathered}
$$

By the changing the argument as $s=t_0+(t-t_0)\,\tau$, from the last estimate we obtain

$$
\begin{aligned}
|\, x_2(t,\,\omega) - x_1(t,\,\omega)\,| &\le \frac{|x_0|}{\alpha} L_0\left(2 + M_0\left(L_{01} + L_{02}\right)\right) \times \\
&\times M_0 \cdot C_0^2 \int\limits_{t_0}^{t} (t-t_0)^{\alpha-1}(1-\tau)^{\alpha-1}(t-t_0)^{\alpha}\tau^{\alpha}(t-t_0)d\tau \le \\
&\le \frac{\Gamma^2(\alpha)}{\Gamma(2\alpha+1)} |x_0| L_0\left(2 + M_0\left(L_{01} + L_{02}\right)\right) M_0 \left[C_0 \cdot (t-t_0)^{\alpha}\right]^2 .
\end{aligned}
\tag{17}
$$

Analogously, taking the estimate (17) into account, for the next difference we derive

$$
\begin{aligned}
&|x_3(t,\omega) - x_2(t,\omega)| \le \\
&\le L_0 \int\limits_{t_0}^{t} \left|(t-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\omega(t-s)^{\alpha}\right)\right| \left[|x_2(s,\omega) - x_1(s,\omega)| + \right.\\
&+ |\max\left\{x_2(\theta,\omega)|\theta \in \left[q_1\left(s, x_2(s,\omega)\right); q_2\left(s, x_2(s,\omega)\right)\right]\right\} - \\
&- \max\left\{x_1(\theta,\omega)|\theta \in \left[q_1\left(s, x_2(s,\omega)\right); q_2\left(s, x_2(s,\omega)\right)\right]\right\}| + \\
&+ |\max\left\{x_1(\theta,\omega)|\theta \in \left[q_1\left(s, x_2(s,\omega)\right); q_2\left(s, x_2(s,\omega)\right)\right]\right\} - \\
&- \left.\max\left\{x_1(\theta,\omega)|\theta \in \left[q_1\left(s, x_1(s,\omega)\right); q_2\left(s, x_1(s,\omega)\right)\right]\right\}|\right] ds \le \\
&\le L_0\left(2 + M_0\left(L_{01} + L_{02}\right)\right) C_0 \int\limits_{t_0}^{t} (t-s)^{\alpha-1} |x_2(s,\omega) - x_1(s,\omega)| ds \le \\
&\le \frac{\Gamma^2(\alpha)}{\Gamma(2\alpha+1)} |x_0| M_0 \left[L_0\left(2 + M_0\left(L_{01} + L_{02}\right)\right)\right]^2 C_0^3 \int\limits_{t_0}^{t} (t-s)^{\alpha-1}(s-t_0)^{2\alpha} ds \le \\
&\le \frac{\Gamma^3(\alpha)}{\Gamma(3\alpha+1)} |x_0| M_0 \left[L_0\left(2 + M_0\left(L_{01} + L_{02}\right)\right)\right]^2 \left[C_0 \cdot (t-t_0)^{\alpha}\right]^3 .
\end{aligned}
\tag{18}
$$

Continuing the estimation processes (15)–(18), for arbitrary difference we obtain

$$
\begin{aligned}
&|\, x_n(t,\,\omega) - x_{n-1}(t,\,\omega)\,| \le \\
&\le \frac{\Gamma^n(\alpha)}{\Gamma(n\alpha+1)} |x_0| M_0 \left[L_0\left(2 + M_0\left(L_{01} + L_{02}\right)\right)\right]^{n-1} \left[C_0 \cdot (t-t_0)^{\alpha}\right]^n .
\end{aligned}
\tag{19}
$$

For the absolute value $|\, x_n(t) - x_{n-1}(t)\,|$ we show that $\sum\limits_{n=1}^{\infty} |\, x_n(t,\,\omega) - x_{n-1}(t,\,\omega)\,| < \infty$ in the space $C\,(t_0;\, T)$. So, the right-hand side of (19) we denote as

$$a_n = \frac{\Gamma^n(\alpha)}{\Gamma(n\alpha+1)} \cdot \left[L_0\left(2+M_0\left(L_{01}+L_{02}\right)\right)\right]^{n-1} \left[C_0 \cdot (t-t_0)^{\alpha}\right]^n .$$

We set

$$a_{n+1} = \frac{\Gamma^{n+1}(\alpha)}{\Gamma((n+1)\alpha+1)} \cdot \left[L_0\left(2+M_0\left(L_{01}+L_{02}\right)\right)\right]^{n} \left[C_0 \cdot (t-t_0)^{\alpha}\right]^{n+1} .$$

Then we consider the following limit

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \\ = L_0\left(2+M_0\left(L_{01}+L_{02}\right)\right)\Gamma(\alpha) C_0 \cdot (t-t_0)^{\alpha} \lim_{n\to\infty} \frac{\Gamma(n\alpha+1)}{\Gamma((n+1)\alpha+1)}. \tag{20}$$

Taking following known formula [24]

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} = z^{a-b}\left[1+\frac{(a-b)(a-b-1)}{2z}+O(z^{-2})\right]$$

into account, we obtain

$$\lim_{n\to\infty} \frac{\Gamma(n\alpha+1)}{\Gamma((n\alpha+\alpha+1)} = \\ = \lim_{n\to\infty} (n\alpha)^{1-\alpha-1}\left[1+\frac{(1-\alpha-1)(1-\alpha-1-1)}{2n\alpha}+O(n\alpha)^{-2}\right] = \\ = \frac{1}{\alpha^{\alpha}} \lim_{n\to\infty} \frac{1}{n^{\alpha}}\left[1+\frac{\alpha(1+\alpha)}{2n\alpha}+O(n\alpha)^{-2}\right] = 0.$$

Consequently, for (20) we have

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = L_0\left(2+M_0\left(L_{01}+L_{02}\right)\right)\Gamma(\alpha)C_0\cdot(t-t_0)^{\alpha}\cdot$$
$$\cdot\lim_{n\to\infty}\frac{\Gamma(n\alpha+1)}{\Gamma((n+1)\alpha+1)} =$$
$$= L_0\left(2+M_0\left(L_{01}+L_{02}\right)\right)\Gamma(\alpha)C_0\cdot(t-t_0)^{\alpha}\cdot\frac{1}{\alpha^{\alpha}}\times$$
$$\times\lim_{n\to\infty}\frac{1}{n^{\alpha}}\left[1+\frac{\alpha(1+\alpha)}{2n\alpha}+O(n\,\alpha)^{-2}\right]=0.$$

Hence, according to d'Alembert's convergence criterion of series, we have

$$\sum_{n=1}^{\infty}\left|\,x_n(t,\,\omega)-x_{n-1}(t,\,\omega)\,\right|\le$$
$$\le\sum_{n=1}^{\infty}\frac{\Gamma^n(\alpha)}{\Gamma(n\alpha+1)}\cdot C_0^n\left[L_0\left(2+M_0\left(L_{01}+L_{02}\right)\right)\right]^{n-1}(t-t_0)^{n\,\alpha}<\infty \quad (21)$$

for all $t\ge t_0$. Since we consider the solution of the integral equation (13) in the space of continuous functions $C\,[t_0;\,T]$, it follows from the (21) that the sequence of functions $\{x_k(t)\}_{k=1}^{\infty}$ converges absolutely and uniformly to solution of the integral equation (13) with respect to argument t. Hence implies the existence of a solution of the problem (1), (2) on the interval $(t_0;\,T)$. Now we show the uniqueness of this solution. Assuming that the integral equation (13) has two different solutions $x(t,\,\omega)$ and $y(t,\,\omega)$ on the interval $[t_0;\,T]$, we obtain the following integral inequality

$$|x(t,\,\omega)-y(t,\,\omega)\,|\le L_0\left(2+M_0\left(L_{01}+L_{02}\right)\right)\times$$
$$\times\int_{t_0}^{t}\left|\,(t-s)^{\alpha-1}E_{\alpha,\,\alpha}\left(-\omega\,(t-s)^{\alpha}\right)\right|\cdot|x(s,\,\omega)-y(s,\,\omega)|\;d\,s. \quad (22)$$

Applying Gronwall–Bellman inequality to estimate (22), we obtain that there holds identity $\|\,x\,(t,\,\omega)-y\,(t,\,\omega\,\|\equiv 0$ for all $t\in[t_0;\,T]$. Therefore, the Cauchy type problem (1), (2) has a unique solution on the interval $(t_0;\,T)$. The Theorem 1 is proved. □

4. Continuous dependence of the solution from parameter ω

In this section we use the following generalized Gronwall–Bellman inequality.

LEMMA 2. *[26]. If for the functions* $0 \leq u(t)$, $f(t)$, $\alpha(t)$, $\beta(t) \in C(t_0; T)$ *is true the inequality*

$$u(t) \leq f(t) + \alpha(t) \int\limits_{t_0}^{t} \beta(s) \cdot u(s)\, d\, s,$$

then the following estimate holds

$$u(t) \leq f(t) + \alpha(t) \int\limits_{t_0}^{t} \beta(s)\, f(s)\, \exp\left\{ \int\limits_{t_0}^{t} \beta(\theta)\, \alpha(\theta)\, d\,\theta \right\} d\, s.$$

Now, we show that the solution $x(t, \omega)$ of the initial value problem for fractional differential equation (1) is stable with respect to a given parameter ω.

THEOREM 2. *Suppose that all the conditions of theorem 1 are fulfilled. Then, the solution of the problem* (1), (2) *on the interval* $(t_0; T)$ *is stable with respect to given parameter* ω.

PROOF. Let $x(t, \omega_1)$ and $x(t, \omega_2)$ be two different solutions of the integral equation (13) corresponding to two different values of the parameter ω_1 and ω_2, respectively. We put: $|\,\omega_1 - \omega_2\,| < \delta$, where $0 < \delta$ is sufficiently small real number.

In the proof of this theorem we use the following inequality

$$\left| E_{\alpha,\alpha}\left(-\omega_1 (t-s)^{\alpha}\right) - E_{\alpha,\alpha}\left(-\omega_2 (t-s)^{\alpha}\right) \right| \leq C_1 \left|\omega_1 - \omega_2\right|, 0 < C_1 = \text{const}.$$

Therefore, similarly to estimate (22), from integral equation (13) we obtain the estimate

$$\begin{aligned}
&\left|(t-t_0)^{1-\gamma}\left(x(t,\omega_1)-x(t,\omega_2)\right)\right| \le \\
&\qquad \le |x_0|\cdot\left|E_{\alpha,\gamma}\left(-\omega_1(t-t_0)^{\alpha}\right)-E_{\alpha,\gamma}\left(-\omega_2(t-t_0)^{\alpha}\right)\right|+ \\
&+\int\limits_{t_0}^{t}(t-s)^{\alpha-1}(t-t_0)^{1-\gamma}\left[\left|E_{\alpha,\alpha}\left(-\omega_1(t-s)^{\alpha}\right)-E_{\alpha,\alpha}\left(-\omega_2(t-s)^{\alpha}\right)\right|\times\right. \\
&\times\left|f\left(s,x(s,\omega_1),\max\left\{x(\theta,\omega_1)|\theta\in[q_1(s,x(s,\omega_1));q_2(s,x(s,\omega_1))]\right\}\right)\right|+ \\
&\left.+L_0\left(2+M_0\left(L_{01}+L_{02}\right)\right)\left|E_{\alpha,\alpha}\left(-\omega_2(t-s)^{\alpha}\right)\right|\cdot\left|x(s,\omega_1)-x(s,\omega_2)\right|\right]ds\le \\
&\le |x_0|\,C_1\cdot|\omega_1-\omega_2|+(t-t_0)^{1-\gamma}C_1\cdot|\omega_1-\omega_2|\,M_0\left|\int\limits_{t_0}^{t}(t-s)^{\alpha-1}ds\right|+ \\
&\qquad +(t-t_0)^{1-\gamma}L_0\left(2+M_0\left(L_{01}+L_{02}\right)\right) \\
&\qquad C_1\int\limits_{t_0}^{t}\left|(t-s)^{\alpha-1}\right|\cdot\left|x(s,\omega_1)-x(s,\omega_2)\right|ds\le \\
&\le M_1(t)\cdot|\omega_1-\omega_2|+M_2(t)\int\limits_{t_0}^{t}\left|(s-t_0)^{\alpha-1}\right|\cdot\left|x(s,\omega_1)-x(s,\omega_2)\right|d\,s, \qquad (23)
\end{aligned}$$

where

$$\begin{aligned}
M_1(t)&=C_1\left[|x_0|+M_0\frac{(t-t_0)^{1+\alpha-\gamma}}{\alpha}\right], \\
M_2(t)&=(t-t_0)^{1-\gamma}L_0\left(2+M_0\left(L_{01}+L_{02}\right)\right)C_1.
\end{aligned}$$

Appling to the estimate (23) the generalized Gronwall–Bellman inequality in Lemma 2, we obtain

$$\begin{aligned}
&\left|(t-t_0)^{1-\gamma}\left(x(t,\,\omega_1)-x(t,\,\omega_2)\right)\right|\le M_1(t)\cdot|\omega_1-\omega_2|+ \\
&\qquad +M_2(t)\int\limits_{t_0}^{t}\left|(s-t_0)^{\alpha-1}\right|\cdot M_1(s)\cdot \\
&\qquad\qquad \cdot|\omega_1-\omega_2|\,\exp\left\{\int\limits_{t_0}^{s}\left|(\xi-t_0)^{\alpha-1}\right|M_2(\xi)\,d\,\xi\right\}d\,s.
\end{aligned}$$

Hence, we have

$$\left|(t-t_0)^{1-\gamma}\left(x(t,\,\omega_1)-x(t,\,\omega_2)\right)\right| \leq B(t)\cdot\left|\omega_1-\omega_2\right|, \tag{24}$$

where

$$B(t)=M_1(t)+M_2(t)\int\limits_{t_0}^{t}\left|(s-t_0)^{\alpha-1}\right|\cdot$$

$$\cdot\, M_1(s)\cdot\exp\left\{\int\limits_{t_0}^{s}\left|(\xi-t_0)^{\alpha-1}\right| M_2(\xi)\,d\,\xi\right\}d\,s.$$

If we put $\varepsilon=\delta\cdot\max\limits_{t_0\leq t\leq T}B\,(t)$, then we obtain from (24) that

$$\left|(t-t_0)^{1-\gamma}\left(x(t,\,\omega_1)-x(t,\,\omega_2)\right)\right| < \varepsilon. \tag{25}$$

From the estimate (25) we see that the solution of the problem (1), (2) is continuous dependence from parameter ω on the interval $(t_0;\,T)$. The theorem 2 is proved. □

5. Conclusion

In this paper we consider the questions of unique solvability of initial value problem for a nonlinear fractional differential equation (1) with nonlinear maxima on the given interval $(t_0;\,T)$. This initial value problem we reduce to the fractional order nonlinear integral equation of Volterra type (3) with nonlinear maxima. The equation (3) has weak singularity at the point $t=t_0$. So, both sides of this integral equation (3) we multiple by the quantity inverse of singularity. Then we use the method of successive approximation and proved the theorem on existence and uniqueness of solution of this problem. We prove also the continuous dependence of the solution of the problem (1), (2) from parameter ω on the interval $(t_0;\,T)$.

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