UDC 517.98

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On Bergman-type spaces in tubular domains over symmetric cones

Some properties of new Bergman-type integral operators on product of the tube domains are obtained. An integral condition in tube to get atomic-type decomposition in multifunctional analytic Bergman spaces in tube domains over symmetric cones are provided.

Bibliography: 6 items.

Получены некоторые свойства новых интегральных операторов типа Бергмана на произведении трубчатых областей. Приведено интегральное условие в трубке для получения разложения атомарного типа в многофункциональных аналитических пространствах Бергмана в трубчатых областях над симметричными конусами.

Библиография: 6 названий.

Keywords: Bergman-type integral operators, Tube domains, Analytic functions, Bergman-type spaces.

Ключевые слова: Интегральные операторы типа Бергмана, Трубные области, Аналитические функции, Пространства типа Бергмана.

1. Introduction

The main goal of this paper is to continue our earlier investigation (see [1]) on Bergman-type multifunctional spaces, related to analytic spaces on product of tube domains and Bergman-type integral operators on them.

⁰The work was supported by the Ministry of Education and Science of the Russian Federation (grant 1.1704.2014K)

We define new Bergman-type spaces in product domains T_Ω^m and apply Whitney-type decomposition of cone and tubular domain in studying of Bergman-type operators on them.

We also provide an integral condition in tube to get atomic-type decomposition in multifunctional analytic Bergman-spaces in tube domains over symmetric cones. Let T_Ω be the tube domain over symmetric cone, and $H(T_\Omega)$ be the space of all analytic functions in tube (see [2]). We define Bergman spaces for $1 \leq p, q < \infty$, $\gamma > \frac{n}{r} - 1$. Let

$$(A_\gamma^{p,q})(T_\Omega) = \left\{ f \in H(T_\Omega) : \left(\int_\Omega \left(\int_{\mathbb{R}^n} |f(x + iy)|^p dx \right)^{\frac{q}{p}} (\Delta^{\gamma - \frac{n}{r}}(y)) dy \right)^{\frac{1}{q}} < \infty \right\}$$

Replacing A by L and H by L^1 we get as usual known larger spaces of measurable functions in tube T_Ω . Note $(A_\gamma^{p,q}) = \{0\}$ if $\gamma \leq \frac{n}{r} - 1$ (see [2]).

We refer to [2] for existence and properties of $\Delta(y)$ function. We need some known facts in analytic spaces in tube T_Ω for proofs.

TEOPEMA A. (Whitney decomposition of the cone Ω , see [2]). *Let $\delta > 0$. Given $y \in \Omega$ we denote $B_\delta(y) = \{\zeta \in \Omega : d(\zeta, y) < \delta\}$; where $d : (\Omega \times \Omega) \rightarrow (R_+)$ is a distance function (see [2]). There exist a sequence of points of Ω such that the following properties hold*

- 1) *The balls $B_\delta(\zeta_k)$ are pairwise disjoint;*
- 2) *the balls $B_\delta(\zeta_k)$ form a covering of Ω ;*
- 3) *there is an integer $N = N(\Omega)$ such that every $y \in \Omega$ belongs to at most N balls $B_\delta(\zeta_k)$ (finite overlapping property).*

As a consequence we have [2] a natural covering of the cone and R_+ with invariant balls and the dyadic decomposition of $(0, \infty)$, $(2^{j-1}, 2^{j+1}) = [B_{\log 2}(2^j)] = \{p \in \Omega : d(p, 2^j) < \log 2\}$, $j \in \mathbb{Z}$ (see [2]).

Let $B_\delta(z_j) = B_j$ be the Bergman ball in tube (see [2]).

TEOPEMA B. (Whitney decomposition of tube, see [2]). *Given $\delta \in (0, 1]$ there exist a sequence $\{z_j\}$ of points of T_Ω such that if $B_j = B_\delta(z_j) : B'_j = B_{\delta/3}(z_j)$*

- 1) the balls (B'_j) are pairwise disjoint;
- 2) the balls (B_j) form a cover of T_Ω ;
- 3) there exists a positive integer $N = N(\Omega)$ (independent of δ) such that every point of T_Ω belongs to at most N balls B_j .

THEOREM C. (see [2]). Let $\gamma > \frac{n}{r} - 1$; $1 \leq p, q < \infty$. Then for every lattice $\{y_i\}$ in Ω there exist constant $c > 0$ such that

$$\frac{1}{c} (\|f\|_{A_\gamma^{p,q}}) \leq \left(\sum_j (\Delta^\gamma(y_j)) \|f(x + iy_j)\|_p^q \right)^{\frac{1}{q}} \leq c (\|f\|_{A_\gamma^{p,q}}), \forall f \in A_\gamma^{p,q}(T_\Omega)$$

where the norm in the middle is the Hardy space norm.

The sequence of points $\{z_i\}$ is called a δ lattice of T_Ω , we have also (see [2])

$$v_\alpha(B_j) \asymp v_\alpha(B'_j) \asymp \Delta^{(\frac{2n}{r})+\alpha}(Im z_j), \alpha > \frac{n}{r} - 1; z_j \in T_\Omega, j = 1, \dots, n.$$

$dv_\alpha(w) = [\Delta^{\alpha-\frac{n}{r}}(v)]dudv$ is a standard Lebesgue measure in tube and weighted Lebesgue measure in tube.

The weighted Bergman kernel B_ν of T_Ω is given by

$$B_\nu(w, z) = (d_\nu) \Delta\left(\frac{w - \bar{z}}{i}\right)^{-\nu-\frac{n}{r}}, d_\nu = (c_\nu^{-1}) \Gamma(\nu + \frac{n}{r}).$$

LEMMA 1 (estimates for Δ^t function and Bergman kernel, see [2]).

- 1) Let $\lambda > \frac{n}{r} - 1$ be fixed. Then $\Delta(y + y') \geq \Delta(y)$; $\forall y, y' \in \Omega$, $|\Delta^{-\lambda}(\frac{x+iy}{i})| \leq (\Delta(y)^{-\lambda})$; $\forall x \in \mathbb{R}^n; y \in \Omega$.
- 2) Let α, β real, then

$$I_{\alpha,\beta}(t) = \int_\Omega (\Delta^\alpha(y+t)) (\Delta^\beta(y)) dy < \infty$$

if $\beta > -1$, $\alpha + \beta < (-\frac{2n}{r} + 1)$, and

$$I_{\alpha,\beta}(t) = (c_{\alpha,\beta}) \Delta^{\alpha+\beta+\frac{n}{r}}(t) \quad (1)$$

Moreover

$$I_\alpha(y) = \int_{\mathbb{R}^n} |\Delta^{-\alpha}(\frac{x+iy}{i})| dx < \infty$$

if $\alpha > \frac{2n}{r} - 1$; and

$$I_\alpha(y) = c_\alpha(\Delta^{-\alpha+\frac{n}{r}}(y)) \quad (2)$$

where $y \in \Omega$.

3) For $\beta > -1$, $\alpha > \frac{n}{r} - 1$, $z = x + iy$, $w = u + iv$, $w, z \in T_\Omega$ we have

$$\int_{T_\Omega} (\Delta^\beta(y)) |B_{\alpha+\beta+\frac{n}{r}}(z, w)| dv(z) \leq c \Delta^{-\alpha}(v), v \in \Omega$$

LEMMA 2 (Reproducing formulas, (see [2])). For any analytic function from $A_\alpha^2(T_\Omega)$ the following integral formula is valid

$$f(z) = \tilde{c}_\alpha \int_{T_\Omega} B_\alpha(z, w)(f(w)) dv_\alpha(w), z \in T_\Omega \quad (3)$$

Let $1 \leq p < \infty$, $1 \leq q < \infty$; $(n/r) \leq p_1$; $\frac{1}{p_1} + \frac{1}{p} = 1$; $(\frac{n}{r} - 1) < \gamma$.

Let $f \in A_\gamma^{p,q}$ then (3) with $\alpha > (\frac{n}{r} - 1)$ is valid (Bergman representation formula with α index is valid).

LEMMA 3 (atomic decomposition of A_γ^p , (see [2])). Let $p \geq 1$, and $\nu > \frac{n}{r} - 1$. Let $\{z_j\}$ be a δ -lattice in T_Ω ; $\delta \in (0, 1)$; $z_j = x_j + iy_j$, $z_j \in T_\Omega$, $j = 1, \dots, r$. Then

$$\|f\|_{A_\nu^p} \asymp \sum_j |f(z_j)|^p \Delta^{\nu+\frac{n}{r}}(y_j).$$

Assume that Bergman projection P_ν is bounded on A_ν^p and let $\{z_j\}$ be a δ -lattice in T_Ω . Then if $f \in A_\nu^p$ then

$$f(z) = \sum_j \lambda_j B_\nu(z, z_j) \Delta^{\nu+\frac{n}{r}}(y_j), z \in T_\Omega; \quad (4)$$

$$\sum_{j=1}^{\infty} |\lambda_j|^p \Delta^{\nu+\frac{n}{r}}(y_j) \leq c \|f\|_{A_\nu^p}^p \quad (5)$$

If

$$\sum_{j=1}^{\infty} |\lambda_j|^p (\Delta^{\nu+\frac{n}{r}}(y_j)) < \infty,$$

then the "sum with B_ν " (4) converges in A_ν^p and the reverse to (5) is true also.

2. Main results

In this section we provide our main theorems. Let $f \in H(T_\Omega \times \cdots \times T_\Omega)$; $f = (z_1, \dots, z_m)$, $m > 1$; this means f function is holomorphic separately by each variable (z_j) . Let $\vec{f} = (f_1, \dots, f_m)$; $f = \prod_{j=1}^m f_j$

$$\|\vec{f}\|_{B_\nu^p}^p = \int_{T_\Omega} |f_1(z)|^p \dots |f_m(z)|^p \Delta^{\nu_1-\frac{n}{r}}(y) \dots \Delta^{\nu_m-\frac{n}{r}}(y) dx dy < \infty;$$

$y = Im z$; $f_j \in H(T_\Omega)$, $1 \leq p < \infty$, $\nu_j > \frac{n}{r} - 1$, $j = 1, \dots, m$. The following question is natural. We assume $\|\vec{f}\|_{B_\nu^p}^p < \infty$. Then can we say that for each (f_j) , $j = 1, \dots, m$ the conclusion of theorem C and lemma 3 is valid. We solve this problem partially in theorem 2.

Note this is true if $m = 1$ by theorem C. Note also the answer for this problem is positive for bounded domains (namely unit ball)(see [3]).

Next let $1 \leq p < \infty$, $f = f(z_1, \dots, z_m)$, we consider analytic subspaces of $H(T_\Omega^m)$, $T_\Omega^m = T_\Omega \times \cdots \times T_\Omega$, $\nu_j > \frac{n}{r} - 1$, $\nu > \frac{n}{r} - 1$, $j = 1, \dots, m$. These are spaces $(A_\nu^p)_1$, $(A_\nu^p)_2$, $(A_\nu^p)_3$ with norms

$$\|f\|_{(A_\nu^p)_1}^p = \int_{T_\Omega} \cdots \int_{T_\Omega} |f(x_1 + iy_1, \dots, x_m + iy_m)|^p \prod_{j=1}^m \Delta^{\nu_j-\frac{n}{r}}(y_j) dx_j dy_j < \infty$$

$$\|f\|_{(A_\nu^p)_2}^p = \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \int_{\Omega} |f(x_1 + iy, \dots, x_m + iy)|^p \Delta^{\nu-\frac{n}{r}}(y) \left(\prod_{j=1}^m dx_j \right) dy < \infty$$

$$\|f\|_{(A_\nu^p)_3}^p = \int_{\mathbb{R}^n} \int_{\Omega} \cdots \int_{\Omega} |f(x + iy_1, \dots, x + iy_m)|^p \prod_{j=1}^m \Delta^{\nu_j-\frac{n}{r}}(y_j) dx dy_j < \infty$$

The natural problem which obviously arise here is to find some relation between these norms and to find appropriate embeddings between these spaces $(A_{\mathcal{V}}^p)_1, (A_{\mathcal{V}}^p)_2, (A_{\mathcal{V}}^p)_3$. Note in the unit disk this problem was solved in [4], [5].

We formulate first some new results on Bergman-type integral operators now.

THEOREM 1. *Let*

$$[R_{x,y}(g)(w)] = \left[(Im\ w)^{-m\frac{2n}{r} + \sum_{j=1}^m y_j} \right] \times \\ \times \int_{T_{\Omega}} \cdots \int_{T_{\Omega}} g(z_1, \dots, z_m) \frac{[\prod_{j=1}^m (Im\ z_j)^{x_j}] dv(z_1) \dots dv(z_m)}{\prod_{j=1}^m |\Delta^{x_j+y_j}(\frac{w-z_j}{i})|}$$

for $g \in L^1(T_{\Omega}^m, dv(z_1) \dots dv(z_m))$, $w \in T_{\Omega}$. Let $1 \leq p < \infty$, $s_j > (\frac{n}{r} - 1)$, $j = 1, \dots, m$. Let also for each $j = 1, \dots, m$, $x_j > x_0$, $y_j > y_0$, where x_0, y_0 are large enough positive numbers. Than there is a constant $c > 0$ such that

$$\int_{T_{\Omega}} |(R_{x,y})(w)|^p (Im\ w)^{(m-1)(\frac{2n}{r}) + \sum_{j=1}^m (s_j - \frac{n}{r})} dv(w) \leq \\ \leq c \int_{T_{\Omega}} \cdots \int_{T_{\Omega}} |g(z_1, \dots, z_m)|^p \prod_{j=1}^m (Im\ z_j)^{s_j - \frac{n}{r}} dv(z_j)$$

Remark 1. Theorem 1 in case of unit ball of C^n were proved in [4].

Remark 2. Note the "reverse operator"

$$(T_{\vec{\beta}} \vec{f})(z) = \int_{T_{\Omega}} \frac{f(z) \Delta^{\frac{1}{m} \sum_{j=1}^m \beta_j - \frac{n}{r}} (Im\ z) dv(z)}{\prod_{j=1}^m \Delta^{\frac{1}{m}(\frac{n}{r} + \beta_j)}(\frac{z_j - \bar{z}}{i})};$$

$z_j \in T_{\Omega}$, $\beta_j > \frac{n}{r} - 1$, $j = 1, \dots, m$ and it is properties were considered recently in [1].

THEOREM 2. *Let $\nu_k > \frac{n}{r} - 1$, $k = 1, \dots, m$, $m \in \mathbb{N}$, $m > 1$. Let $1 \leq p < \infty$, let for some fixed big enough β_0 and all $\beta > \beta_0$, and $z_j \in T_{\Omega}$; $f_j \in H(T_{\Omega})$, $j = 1, \dots, m$.*

$$f_1(z_1) \dots f_m(z_m) = \tilde{c}_{\beta} \int_{T_{\Omega}} \frac{f(z) \Delta^{\frac{1}{m} \sum_{j=1}^m \beta_j - \frac{n}{r}} (Im\ z) dv(z)}{\prod_{j=1}^m \Delta^{\frac{1}{m}(\frac{n}{r} + \beta_j)}(\frac{z_j - \bar{z}}{i})},$$

where $\tilde{c}_\beta$ is a constant of reproducing formula from lemma 2.

Let also

$$\int_{T_\Omega} \prod_{k=1}^m |f_k(z)|^p \Delta^{(m-1)\frac{n}{r} + \sum_{k=1}^m \nu_k - \frac{n}{r}}(Im\ z) dv(z) < \infty,$$

Then for each f_k the conclusion of theorem C for $p = q$ and lemma 3 are valid.

Remark 3. The complete analogue of theorem 2 for case of unit ball in C^n is valid (see [3]).

Let $\vec{\alpha} = (\alpha_1, \dots, \alpha_m)$, $\vec{\beta} = (\beta_1, \dots, \beta_m)$ or $\vec{\beta} = (\beta, \dots, \beta)$.

We modify the $R_{x,y}$ operator we defined above.

Let also for $g \in L^1(T_\Omega^m)$

$$[G_{\vec{\alpha}, \beta}(g)](x + iy_1, \dots, x + iy_m) = \int_{T_\Omega} \frac{g(w) [\Delta^\beta(Im\ w)] dv(w)}{\left[\prod_{j=1}^m \Delta^{\alpha_j} \left(\frac{w - (\bar{x} + i y_j)}{i} \right) \right]},$$

$x \in \mathbb{R}^n$, $y_j \in \Omega$, $j = 1, \dots, m$, $\alpha_j > 0$, $\beta > \frac{n}{r} - 1$, $j = 1, \dots, m$.

$$[\tilde{G}_{\vec{\alpha}, \vec{\beta}}(g)](x_1 + iy, \dots, x_m + iy) = \int_{T_\Omega} \frac{g(w) [\Delta^\beta(Im\ w)] dv(w)}{\left[\prod_{j=1}^m \Delta^{\alpha_j} \left(\frac{w - (\bar{x}_j + i y)}{i} \right) \right]},$$

$$\beta > \frac{n}{r} - 1, j = 1, \dots, m; \alpha_j > 0,$$

$x_j \in \mathbb{R}^n$, $j = 1, \dots, m$, $y \in \Omega$.

The analogue of following theorem in the unit disk can be found in [5].

A complete proof of the following theorem can be found in [6].

In our theorem below all parameters are positive numbers.

THEOREM 3. For $1 \leq p < \infty$, some $\alpha_j \in (\alpha_0, \alpha'_0)$; $\beta_j \in (\beta_0, \beta'_0)$, $\beta \in (\tilde{\beta}_0, \tilde{\beta}'_0)$, $\nu_j > (\frac{n}{r} - 1)$, $\tau_j > (\frac{n}{r} - 1)$, $j = 1, \dots, m$, for some fixed positive α_0^j , $(\alpha_0^j)'$, β_0^j , $(\beta_0^j)'$, $\tilde{\beta}_0^j$, $(\tilde{\beta}_0^j)'$, $j = 1, \dots, m$.

The following estimates are valid

$$1) \|G_{\vec{\alpha}, \beta}(g)\|_{(A_{\vec{\nu}}^p)_3} \leq c_1 \|g\|_{(A_\tau^p)(T_\Omega)}; \text{ for some values } \vec{\nu} \text{ and } \tau.$$

$$2) \|\tilde{G}_{\vec{\alpha}, \beta}(g)\|_{(A_{\vec{\nu}}^p)_2} \leq c_2 \|g\|_{(A_\tau^p)(T_\Omega)}; \text{ for some values } \nu \text{ and } \tau.$$

3) $\|V_{\vec{\alpha}, \vec{\beta}}(g)\|_{(A_{\vec{\nu}}^p)_1} \leq c_3 \|g\|_{(A_{\vec{\tau}}^p)_3(T_\Omega)}$; for some values ν and $\vec{\tau}$.

4) $\|U_{\vec{\alpha}, \vec{\beta}}(g)\|_{(A_{\vec{\nu}}^p)_1} \leq c_4 \|g\|_{(A_{\vec{\tau}}^p)_2(T_\Omega)}$; for some values ν and $\vec{\tau}$.

where $\alpha_0^j, \dots, (\beta_0^j)'$ depend on $\nu_j, \tau_j, p, n, \nu, \tau, j = 1, \dots, m$ and $\frac{1}{p} + \frac{1}{p'} = 1$.

Remark 4. For less general case where $\mathbb{R}^n$ is unit interval I , $I = (0, 1)$ and Ω is unit circle $T = \{|z| = 1\}$ such type results were proved previously in [4], [5].

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Received

17.08.2020